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# Efficient Solutions with the LRPS Method for Non-Linear Fractional Order Tuberculosis Models

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**Abstract**

In this research article, we present a novel Non-Linear Fractional Order Tuberculosis mathematical model (NLFOTB) and introduce an efficient technique to obtain its solution. Fractional Order Models (FOMs) have garnered significant attention in contemporary research due to their widespread applicability. We address the challenge of solving the coupled Initial Value Problems (IVPs) associated with NLFOTB models by utilizing the groundbreaking LRPS method, which combines the RPS approach with the Laplace transform operator. This innovative approach generates approximate solutions in rapidly converging series forms, offering enhanced efficiency and reduced computational effort compared to conventional methods. Through the implementation of the LRPS method, we successfully derive an approximate solution for the NLFOTB model, contributing significantly to the field. Furthermore, our proposed approach demonstrates its efficacy in accurately capturing the dynamics of Tuberculosis (TB) through extensive computations and graphical representations, contributing to a deeper understanding of TB dynamics within a mathematical framework. Additionally, the LRPS method shows promise in tackling real world problems involving differential equations of various orders. Future investigations can extend the application of the LRPS method to explore other Fractional Order Models, further validating its effectiveness in a wide range of epidemic scenarios. Consequently, our study not only provides valuable insights into Tuberculosis dynamics but also introduces a powerful computational tool applicable to various practical problems in diverse disciplines, making a substantial contribution to the field of mathematical modeling and computation.

**Keywords:** Fractional order Tuberculosis mathematical model, Caputo's derivative operator, Laplace residual power series

## Introduction

Mathematical modeling in epidemiology is a powerful tool for understanding disease spread [1]. Tuberculosis (TB) [2], caused by the Mycobacterium tuberculosis bacillus, remains a formidable global health challenge, being one of the deadliest diseases worldwide. In 2013, approximately 9 million people were infected with TB, resulting in 1.5 million reported deaths, with 360,000 of those being HIV positive. Despite progress, TB incidence has been gradually declining by an average of 1.5% per year between 2000 and 2013, saving an estimated 37 million lives through effective diagnosis and treatment. The global mortality rate from TB has decreased by 45% between 1990 and 2013, accompanied by a 41% drop in prevalence. However, TB-related deaths are still considered unacceptably high, and efforts are focused on accelerating programs to further reduce the burden of TB, aiming to achieve the Millennium Development Goals (MDGs) and the Stop TB Partnership target of a 50% reduction by 2015.

Tuberculosis has significant repercussions on family and social relationships, resulting in adverse health and economic consequences [3,4]. Tuberculosis (TB) is caused by a bacterium called Mycobacterium tuberculosis. The bacteria usually attack the lungs, but TB bacteria can attack any part of the body such as the kidney, spine, and brain. Enhancing case detection and notification rates is crucial for reducing the TB burden in the population. However, several factors can hinder these efforts. Individuals affected by TB, along with their families, often experience stigmatization, which manifests as feelings of shame, blame, and judgment. Such stigma can act as a barrier to improving tuberculosis case detection rates [5], since patients and their families fear negative judgments associated with TB diagnosis.

A survey conducted in Nigeria (a high burden country) examined factors adversely impacting the implementation of the directly observed treatment, short-course (DOTS) strategy, aimed at reducing TB

incidence in the country [5,6]. The survey revealed that many individuals lacked knowledge about TB transmission, signs, symptoms, and government health policies on tuberculosis and treatment. As a consequence, this lack of awareness led to delays in reporting probable TB cases for treatment, thereby increasing the risk of disease transmission. Addressing this knowledge gap through targeted educational initiatives is critical in the fight against tuberculosis.

Fractional derivatives have emerged as a powerful tool for modeling various real-life problems in biology, physics, control theory, and other engineering and scientific fields [7-13]. The utilization of numerical and approximate approaches has facilitated the computation of solutions for fractional differential equations (DEs) [14]. These fractional ordinary and partial differential equations (PDEs) play a pivotal role in diverse scientific and engineering disciplines [15-17]. However, obtaining analytical solutions for systems of fractional order linear and non-linear initial value problems remains a challenging task due to the intricacies involved in computing fractional operators. As a result, there has been a pressing need to advance different techniques to tackle these complexities. Notable examples include the Haar wavelet collocation approach, Adomian decomposition approach, Sumudu transform approach, variational iteration approach, differential transforms approach, and homotopy perturbation coupled with the Sumudu transform approach [17-19]. By exploring and refining these innovative methods, we aim to provide a breakthrough in efficiently obtaining analytical solutions for fractional order tuberculosis mathematical models. This research contributes significantly to the field by enhancing our understanding and analytical capabilities for tackling tuberculosis dynamics, opening new avenues for addressing complex real-world problems in various domains.

The Residual Power Series (RPS) approach, in conjunction with the Laplace transform, is a numerical and analytical technique widely used to compute different types of fractional order ordinary, fuzzy differential equations (DEs), partial DEs, and integro-differential equations. The Laplace transform approach holds a crucial role in solving substantial models across various natural sciences fields [20]. Combining the Residual Power Series with the Laplace transform, the Laplace RPS (LRPS) technique offers several advantages, including minor computational requirements, faster computations, and high accuracy [11,21,22]. This innovative approach provides an efficient and effective solution for handling complex mathematical models in scientific research, with applications across different domains.

In this article, we conducted a comprehensive study on the LRPS strategy applied to fractional order equations. Our focus was on analyzing the LRPS solution's behavior for different fractional orders, and we presented graphs of $S(t)$, $E(t)$, and $I(t)$ over the interval $(0,1]$. The results demonstrated the LRPS method's efficiency and accuracy in approximating solutions and accurately predicting compartmental behavior within the specified range. Comparatively, we found that the LRPS approach offers several advantages over other numerical and analytical methods commonly used in epidemiology. Its sophistication, inclusivity, and efficiency make it an attractive option for generating numerical solutions for interconnected fractional order nonlinear differential equations. The discussions presented in this article have significantly contributed to the field of epidemiology by introducing and showcasing the LRPS approach's effectiveness. As a valuable tool for investigating and validating epidemic models, the LRPS method offers improved efficiency and convenience. We believe that our research will inspire further exploration and utilization of the LRPS technique in solving nonlinear models, contributing to advancements in the field of epidemiology.

## Preliminaries

In this section, we will revisit the key characteristics of Caputo's derivative operator, as discussed in the works of Dutta [23] and the book by Kilbas [7]. Additionally, we will explore the fractional Taylor's formula, a crucial aspect of fundamental theories. This review aims to present important definitions and findings concerning Caputo's fractional derivatives and the fractional Laplace transform.

## Definition

The fractional order derivative for the function $z(v)$ according to Caputo's definition is expressed as follows:

$$D_v^{\gamma} z_i(v) = \frac{1}{\Gamma(n-\gamma)} \int_0^v \frac{z^n(\delta)}{(v-\delta)^{\gamma+1-n}}\, d\delta,\ v > 0 \quad (1)$$

The LRPS approach centers on solving linear and non-linear fractional differential equations (NLFDEs) analytically by determining the coefficients of the fractional power series approximate

solution. The system of differential equations involving fractional orders, as defined by Caputo, can be expressed as follows:

$$D_v^{\gamma} z_i(v) = g_i\big(v, z_1(v), z_2(v), \ldots . z_n(v)\big), v \geq 0, \gamma \in (0,1] \quad (2)$$

with $z_i(0) = a_i, i = 1,2, \ldots ., n$. Here $z_i(v)$ be the unknown smooth functions and $g_i$ be the linear as well non-linear functions.

Firstly, we are taking Laplace transform on both sides of the system as follows:

$$L\big[D_v^{\gamma} z_i(v)\big] = L\big[g_i\big(v, z_1(v), z_2(v), \ldots . z_n(v)\big)\big] \quad (3)$$

employing the formula,

$$L\big[D_v^{\gamma} z_i(v)\big] = s^{\gamma} Z(s) - s^{(\gamma-1)} z(0) \quad (4)$$

by setting $G(s) = L\big[g_i\big(v, z_1(v), z_2(v), \ldots . z_n(v)\big)\big]$ and using the initial conditions, system can be found as:

$$Z_i(s) = \frac{a_i}{s} + \frac{1}{s^{\gamma}} G(s) \quad (5)$$

for $1 \leq i \leq n$. Subsequently, the LSS (Laplace series solution) within the LRPS approach can be represented as follows:

$$Z_n(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n\gamma+1}}, s > 0 \quad (6)$$

Since$a_n = \lim_{s \to 0} s Z_n(s)$. The *j*-th Laplace solution series is given as:

$$Z_n^j(s) = \frac{a_0}{s} + \sum_{n=1}^{\infty} \frac{a_n}{s^{n\gamma+1}}, s > 0 \quad (7)$$

and the *j*-th residual Laplace functions is expressed as following:

$$L\big[Res_j(Z_n(s))\big] = Z_n^j(s) - \frac{a_0}{s} - \frac{1}{s^{\gamma}} G(s) \quad (8)$$

The coefficient $a_n$ can be determined by solving $\lim_{s \to \infty} s^{j\gamma+1} L\big[Res_j\big(Z_i(s)\big)\big] = 0$, for $1 \leq j \leq n$and $0 < \gamma \leq 1$. Now taking inverse Laplace transform to get the *j*-th LFPSS of the problem.

**Fractional Tuberculosis (FTB) model**

The Tuberculosis model utilizing Caputo's approach is presented to us as follows:

$$\begin{aligned} D_t^{\gamma} S^*(t) &= v - \varrho S^*(t) I^*(t) - \varpi S^*(t) \\ D_t^{\gamma} E^*(t) &= (1-\rho)\varrho S^*(t) I^*(t) - \kappa E^*(t) - \varpi E^*(t) \\ D_t^{\gamma} I^*(t) &= \rho\varrho S^*(t) I^*(t) + \kappa E^*(t) - \varpi I^*(t) - \zeta I^*(t) \end{aligned} \quad (9)$$

Constrained by the initial conditions,

$$S^*(0) = S_0^*, E^*(0) = E_0^*, \mathrm{I}^*(0) = I_0^* \quad (10)$$

In this particular model, the variables $S^*$, $E^*$ and $I^*$ are used to represent susceptible, exposed (individuals who are infected but have not been detected by testing) and infectious respectively, at time *t*. Additionally, $\upsilon$ denotes the recruitment rate of susceptible individuals,$\varpi$represents the natural death rate, $\varrho$be the transmission rate of active TB, $\kappa$ is the progression rate from latent TB to active TB (rate of slow progression), $\varsigma$be the death rate due to TB infection, $\rho$ signifies the rate of fast progression. It's important to note that the whole population remains unchanged in this SEI model, as indicated by the equation $S^*(t) + E^*(t) + I^*(t) = N(t)$, where $N(t)$ represents the total population.

**LRPS solution of TB model**

Now we are going to find the approximate solution of FTB model (9) by using LRPS approach. Firstly, taking the Laplace transform on both sides and using the initial conditions, we have the following system:

$$S(s) = \frac{S_0^*}{s} + \frac{v}{s^{\gamma+1}} - \frac{\varrho}{s^\gamma} L[L^{-1}S(s)L^{-1}I(s)] - \frac{\varpi}{s^\gamma} S(s) \quad (11)$$
$$E(s) = \frac{E_0^*}{s} + \frac{(1-\rho)\varrho}{s^\gamma} - L[L^{-1}S(s)L^{-1}I(s)] - \frac{\kappa+\varpi}{s^\gamma} E(s)$$
$$I(s) = \frac{I_0^*}{s} + \frac{\rho\varrho}{s^\gamma} - L[L^{-1}S(s)L^{-1}I(s)] + \frac{\kappa}{S^\gamma} E(s) - \frac{\kappa+\varsigma}{s^\gamma} I(s)$$

We are currently employing the *k*-th LSS assumption for the system (11), which can be expressed as follows:

$$S^k(s) = \frac{S_0^*}{s} + \sum_{n=1}^{k} \frac{a_n}{s^{n\gamma+1}}, s > 0 \quad (12)$$
$$E^k(s) = \frac{E_0^*}{s} + \sum_{n=1}^{k} \frac{b_n}{s^{n\gamma+1}}, s > 0$$
$$I^k(s) = \frac{I_0^*}{s} + \sum_{n=1}^{k} \frac{c_n}{s^{n\gamma+1}}, s > 0$$

Subsequently, the coefficients $a_n$, $b_n$and$c_n$can be found by forming the *k*-th LR-function of the system (11) in the following manner:

$$L[Res_k S(s)] = S^k(s) - \frac{S_0^*}{s} - \frac{v}{s^{\gamma+1}} + \frac{\varrho}{s^\gamma} L[L^{-1}S^k(s)L^{-1}I^k(s)] + \frac{\varpi}{s^\gamma} S^k(s) \quad (13)$$
$$L[Res_k E(s)] = E^k(s) - \frac{E_0^*}{s} - \frac{(1-\rho)\varrho}{s^\gamma} L[L^{-1}S^k(s)L^{-1}I^k(s)] + \frac{\kappa+\varpi}{s^\gamma} E^k(s)$$
$$L[Res_k I(s)] = I^k(s) - \frac{I_0^*}{s} - + \frac{\rho\varrho}{s^\gamma} L[L^{-1}S^k(s)L^{-1}I^k(s)] - \frac{\kappa}{s^\gamma} E^k(s) + \frac{\varpi+\zeta}{s^\gamma} I^k(s)$$

Here for the value of $k$ = 1, 2, 3..., we can solve the system (13). The value of coefficients $a_n$, $b_n$and$c_n$can be computed by using:

$$\lim_{s\to\infty} s^{k\gamma+1} L[Res_k(S(s))] = 0$$
$$\lim_{s\to\infty} s^{k\gamma+1} L[Res_k(E(s))] = 0$$
$$\lim_{s\to\infty} s^{k\gamma+1} L[Res_k(I(s))] = 0$$

Some of the coefficients are:

$a_1 = v - \varrho S_0^* I_0^* - \varpi S_0^*$, $b_1 = (1-\rho)\varrho S_0^* I_0^* - (\kappa+\varpi)E_0^*$, $c_1 = \rho\varrho S_0^* I_0^* + \kappa E_0^* - (\varpi+\varsigma)I_0^*$.
$a_2 = -\varrho c_1 S_0^* - \varrho a_1 I_0^* - \varpi a_1$,
$b_2 = (1-\rho)\varrho S_0^* c_1 + (1-\rho)\varrho I_0^* a_1 + (\kappa+\varpi)b_1$,
$c_2 = S_0^* c_1 \rho\varrho + \rho\varrho a_1 I_0^* + b_1\kappa - (\varpi+\varsigma)c_1$
$a_3 = -\varrho c_2 S_0^* - \varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - a_2\varrho I_0^* - \varpi a_2$
$b_3 = (1-\rho)\varrho S_0^* c_2 + (1-\rho)\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - (1-\rho)\varrho a_2 I_0^* - (\kappa+\varpi)b_2$
$c_3 = \rho\varrho S_0^* c_2 + \rho\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} + \rho\varrho a_2 I_0^* + \kappa b_2 - c_2(\varpi+\varsigma)$ and so on.

The LSS of the system (11) is presented as:

$$S(s) = \frac{S_0^*}{s} + \frac{v - \varrho S_0^* I_0^* - \varpi S_0^*}{s^{\gamma+1}} + \frac{-\varrho c_1 S_0^* - \varrho a_1 I_0^* - \varpi a_1}{s^{2\gamma+1}} + (-\varrho c_2 S_0^* - \varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - a_2\varrho I_0^* - \varpi a_2)\frac{1}{s^{3\gamma+1}} + \cdots \quad (14)$$

$$E(s) = \frac{E_0^*}{s} + \frac{(1-\rho)\varrho S_0^* I_0^* - (\kappa+\varpi)E_0^*}{s^{\gamma+1}} + \frac{(1-\rho)\varrho S_0^* c_1 + (1-\rho)\varrho I_0^* a_1 + (\kappa+\varpi)b_1}{s^{2\gamma+1}} + \left((1-\rho)\varrho S_0^* c_2 + (1-\rho)\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - (1-\rho)\varrho a_2 I_0^* - (\kappa+\varpi)b_2\right)\frac{1}{s^{3\gamma+1}} + \cdots$$

$$I(s) = \frac{I_0^*}{s} + \frac{\rho\varrho S_0^* I_0^* + \kappa E_0^* - (\varpi+\varsigma)I_0^*}{s^{\gamma+1}} + \frac{S_0^* c_1 \rho\varrho + \rho\varrho a_1 I_0^* + b_1\kappa - (\varpi+\varsigma)c_1}{s^{2\gamma+1}} + \left(\rho\varrho S_0^* c_2 + \rho\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} + \rho\varrho a_2 I_0^* + \kappa b_2 - c_2(\varpi+\varsigma)\right) + \frac{1}{s^{3\gamma+1}} + \ldots\ldots$$

Taking inverse Laplace transform on both sides of system (14), to obtain the LRPS solution of the system (9), as

$$S^*(t) = S_0^* + \frac{(\nu - \varrho S_0^* I_0^* - \varpi S_0^*)t^\gamma}{\Gamma(\gamma+1)} + \frac{(-\varrho c_1 S_0^* - \varrho a_1 I_0^* - \varpi a_1)t^{2\gamma}}{\Gamma(2\gamma+1)} + \left(-\varrho c_2 S_0^* - \varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - a_2\varrho I_0^* - \varpi a_2\right)\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots \quad (15)$$

$$E^*(t) = E_0^* + \frac{((1-\rho)\varrho S_0^* I_0^* - (\kappa+\varpi)E_0^*)t^\gamma}{\Gamma(\gamma+1)} + \frac{((1-\rho)\varrho S_0^* c_1 + (1-\rho)\varrho I_0^* a_1 + (\kappa+\varpi)b_1)t^{2\gamma}}{\Gamma(2\gamma+1)} + \left((1-\rho)\varrho S_0^* c_2 + (1-\rho)\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} - (1-\rho)\varrho a_2 I_0^* - (\kappa+\varpi)b_2\right)\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots$$

$$I^*(t) = I_0^* + \frac{(\rho\varrho S_0^* I_0^* + \kappa E_0^* - (\varpi+\varsigma)I_0^*)t^\gamma}{\Gamma(\gamma+1)} + \frac{(S_0^* c_1 \rho\varrho + \rho\varrho a_1 I_0^* + b_1\kappa - (\varpi+\varsigma)c_1)t^{2\gamma}}{\Gamma(2\gamma+1)} + \left(\rho\varrho S_0^* c_2 + \rho\varrho a_1 c_1 \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} + \rho\varrho a_2 I_0^* + \kappa b_2 - c_2(\varpi+\varsigma)\right) + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \ldots\ldots$$

**Numerical results**

We present the numerical results obtained by applying the LRPS technique to solve the FTB model (9). The outcomes vividly illustrate the remarkable performance and efficiency of the LRPS method in handling epidemic models. Utilizing this proposed technique, we successfully derive an approximate solution for the FTB model (9), expressed as a rapidly converging series. These findings underscore the efficacy of the LRPS approach, providing precise and efficient solutions to complex epidemic scenarios. The approximated solutions are visually represented in the form of graphs, illustrating the behavior of the compartments $S^*(t)$, $E^*(t)$ and $I^*(t)$. These graphs provide a concise and effective visualization of the outcomes.

$$\begin{aligned} D_t^\gamma S^*(t) &= 0.03 - 0.35\, S^*(t)I^*(t) - 0.01\, S^*(t) \\ D_t^\gamma E^*(t) &= 0.28\, S^*(t)I^*(t) - 0.01013\, E^*(t) \\ D_t^\gamma I^*(t) &= 0.07\, S^*(t)I^*(t) + 0.00013\, E^*(t) - 0.31\, I^*(t) \end{aligned} \quad (16)$$

The given initial conditions are as follows: $S^*(0) = 11000$, $E^*(0) = 3500$, $I^*(0) = 500$. Furthermore, the estimated parameters have been organized into the **Table 1**. Here, $\gamma$ is the order of fractional derivative described in the Caputo sense where $0 < \gamma \leq 1$. We are using the LRPS technique in model (16) and calculated all series solutions $S^*(t)$, $E^*(t)$ and $I^*(t)$.

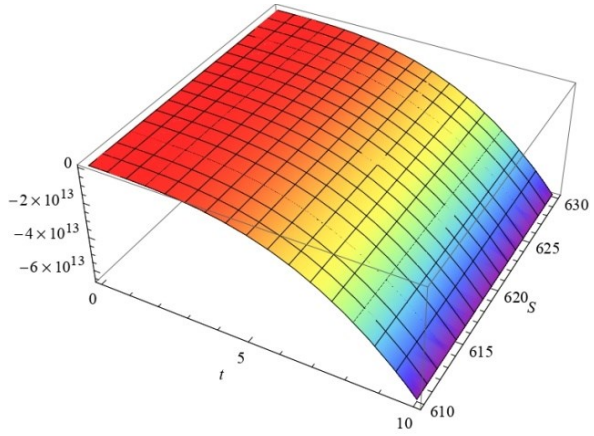


**Figure 1** Plot of LRPS solution of $S^*(t)$.

Additionally, we have transformed the FTB model into the Simple TB model for $\gamma = 1$. The corresponding series solution for $\gamma = 1$ is depicted in **Figures 1 - 3**.

**Table 1** The estimated values of parameters using in model (9).

| Parameters | Estimated values |
|---|---|
| $\upsilon$ | 0.03 |
| $\varrho$ | 0.35 |
| $\varpi$ | 0.01 |
| $\rho$ | 0.2 |
| $\kappa$ | 0.00013 |
| $\varsigma$ | 0.3 |

The LRPS-solution for different fractional orders ($\gamma = 0.2, 0.4, 0.6, 0.8$) within the interval $(0, 1]$. These figures demonstrate the effectiveness of the LRPS strategy, as obtained approximations showcase a remarkable level of efficiency, even with a relatively small number of terms in this particular example.

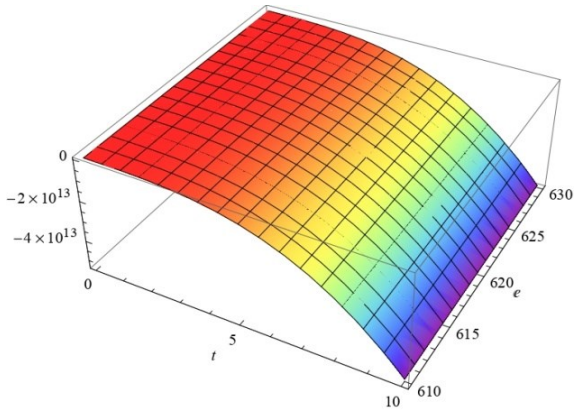


**Figure 2** Plot of LRPS solution of $E^*(t)$.

Notably, the LRPS method proves capable of accurately capturing the dynamics of the TB during the specified time period. It highlights the model's ability to approximate the susceptible, exposed and infected populations, and provides valuable insights into the epidemic's progression. However, it is worth mentioning that further improvements in efficiency can be achieved by increasing the number of terms in the power series, ensuring even more precise predictions and a Finer-grained understanding of the TB dynamics.

Additionally, to analyze the impact of the FTB model, we investigate the LRPS solutions for the exposed, susceptible, and infected populations for various values of $\gamma$ (e.g., $\gamma$ = 0.2, 0.4, 0.6, 0.8). **Figures 4 - 6** illustrate these solutions over the interval (0, 1].

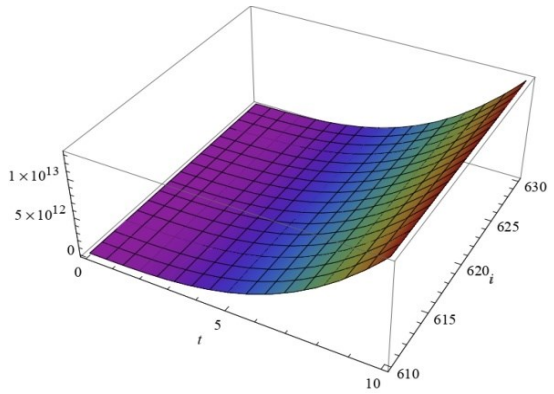


**Figure 3** Plot of LRPS solution of $I^*(t)$.

The graphical representations vividly demonstrate the greater flexibility offered by the fractional derivative in comparison to the integer order derivative.

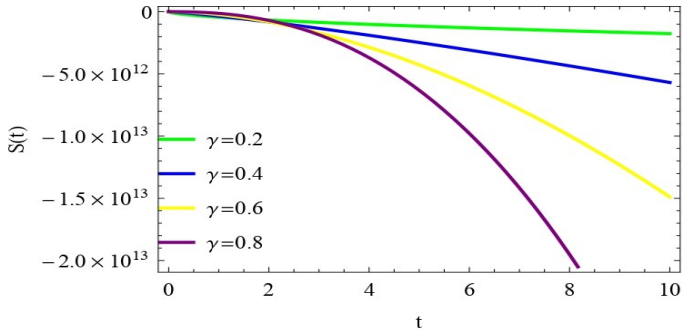


**Figure 4** Plot of $S^*(t)$ for different values of $\gamma$.

Introducing fractional parameters with non-integer values leads to noticeable variations in the compartments of the proposed FTB model. As the fractional order approaches the integer order, the curves of the compartments $S(t)$, $E(t)$, and $I(t)$ in the FTB model converge towards those of the classical TB model, providing valuable insights into the dynamics of TB infection.

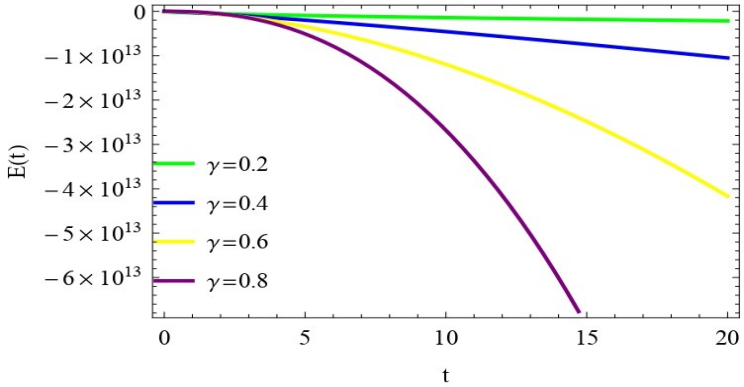


**Figure 5** Plot of $E^*(t)$ for different values of $\gamma$.

The depicted graphs not only to confirm the precision of the approach/method in predicting the dynamics of the compartments $S(t)$, $E(t)$ and $I(t)$ within the specified region but also highlight the exceptional level of agreement among the obtained approximations. This remarkable consistency further reinforces the reliability and robustness of the method's predictions.

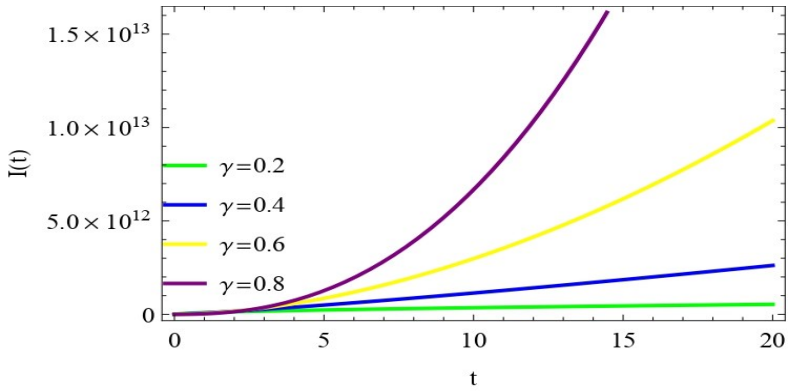


**Figure 6** Plot of $I^*(t)$ for different values of $\gamma$.

## Conclusions

Our paper introduces a novel and effective approach to approximate the solution of FTB model, with a specific focus on Tuberculosis (TB) dynamics. The LRPS technique proposed in this study serves as a valuable tool for researchers in epidemiology, enabling the investigation and validation of epidemic models with improved accuracy. This elegant and efficient method allows for constructing approximate solutions for interconnected NFODEs. Through our research, we have provided valuable insights into the impact of Fractional Order Derivatives on the SEI model, visually depicting the solutions for various values of the fractional order parameter $\gamma$. Notably, we observed a gradual convergence of the FOSEI model's solutions towards those of the classical SEI models as the fractional order approached the integer order. This finding emphasizes the significant influence of the FD on the dynamics of epidemics. To summarize, our proposed approach offers a promising solution for approximating solutions of FODEs within the domain of epidemiology and related fields. It equips researchers with a powerful tool for investigating and comprehending complex epidemic systems. Furthermore, we anticipate that this method will inspire further exploration and application in other nonlinear models and various domains. However, to fully unlock the potential of this approach, additional research is necessary to explore its capabilities and applications comprehensively. By harnessing the advantages of the LRPS technique, we can further advance our understanding of epidemic dynamics and pave the way for more effective interventions and control strategies in the fight against infectious diseases like TB.

## Future directions

Here are several potential avenues for future research in utilizing the LRPS methodology to find solution of SEIR and modified SEIR type epidemic models:

1) Extension to different epidemic models: Further exploration can focus on applying the LRPS approach to different types of epidemic models, such as the SEIRD (including deaths) or SEIRS (including a susceptible-recovered-susceptible loop) models. This would expand the applicability and effectiveness of the LRPS approach in capturing the dynamics of various infectious diseases.

2) Incorporating intervention strategies: In future research, it is essential to explore the influence of intervention strategies, such as vaccination campaigns or social distancing measures. Integrating these strategies into the LRPS approach will allow re- searchers to assess their efficacy in controlling the transmission of infectious diseases and gauge their potential for mitigating future epidemics.

3) Parameter estimation and sensitivity analysis: The utilization of the LRPS approach for parameter estimation and sensitivity analysis offers significant insights into the impact of model parameters on epidemic dynamics. By conducting such analysis, researchers can gain a better understanding of the crucial factors driving disease transmission. This knowledge can play a vital role in making informed decisions regarding disease control and prevention strategies, ultimately improving our ability to tackle infectious diseases effectively.

Overall, exploring these future research directions has the potential to significantly advance and promote the wider adoption of the LRPS approach in the analysis and comprehension of FTB models. Ultimately, this will facilitate the development of robust strategies for disease control and prevention, bolstering our efforts to combat and manage infectious diseases effectively.

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