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The Generalized Distributions on the Unit Interval based on the T-Topp-Leone Family of Distributions

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Abstract

The Topp-Leone (TL) distribution is introduced by Topp and Leone [1]. Its probability density function is a simple function with only one parameter. Even though the TL distribution has been discussed and applied in many research fields, but there is a limitation about its shape. In this article, we propose the T-TL family of distributions using quantile function of T-$R\{Y\}$ family of distributions to generate generalized TL distributions including the Weibull-TL{exponential}, the log-logistic-TL{exponential}, the logistic-TL{extreme value}, the exponential-TL{log-logistic} and the normal-TL{logistic} distributions. Some associated properties and inferences are discussed. Some graphical representations related to the probability density function are shown. Finally, 3 real datasets are applied to illustrate the generalized TL distributions.

Keywords: Topp-Leone distribution, Quantile function, T-R{Y} family of distributions, Maximum likelihood estimation, Kolmogorov-Smirnov test, Shannon entropy, Mean deviation

Introduction

The Topp-Leone (TL) distribution is an attractive probability distribution, its domain can be either finite $(0 < x < b)$ or infinite $(0 < x < b < \infty)$ [1]. If $b = 1$ then the TL distribution is bounded on $(0, 1)$, which provides closed forms of the cumulative distribution function (cdf) and probability density function (pdf). According to one parameter, the estimation part for the TL distribution is not complicated. Nadarajah and Kotz [2] studied some properties of the TL distribution and provided its moments, central moments and characteristic function. Furthermore, there are also numerous authors interested in this distribution. The TL distribution was took to family as generator after Eugene *et al.* [3] presented the beta generated (BG) family of distributions, which is the beta distribution with support on $(0, 1)$. This work was motivated a large number of researchers to develop new family by applying new generators such as the Kumaraswamy generator [4], the generalized beta generator [5], the McDonald generator [6] and the TL generator of distributions [7,8]. Then, there are a large number of distributions generated by the TL family such as the TL generalized exponential distribution [7], the TL inverse Weibull distribution [9], the TL normal distribution [10], and the TL Lomax distribution [11].

Alzaatreh *et al.* [12] extended the BG family to general method called transformed-transformer distribution or T-X family. With their extension, the support of generators become more flexible and we can apply more distributions as generators. Although the extra function seem to be more complicated, in practice, some expertise attracts significant to the most convenient one. In other word, the generators with both closed form of cdf and pdf, which have a few numbers of parameters, provide an easily accessible in deriving the parameter estimation part. Although this method is popular, but it has added a function $W(F(x))$ to satisfy some conditions for developing the family. Later, Aljarrah *et al.* [13] introduced the quantile function instead of the adjusted function called T-$X\{Y\}$ family of distributions. Finally, the T-$X\{Y\}$ was re-defined as T-$R\{Y\}$ family of distributions by Alzaatreh *et al.* [14]. However, using a generator with support lying between 0 and 1 is still convenient. Furthermore, there are many works extending TL family such as the TL odd log-logistic-G family [15], the odd log-logistic TL-G family [16] and the TL exponentiated-G family [17].

We can notice that the popular unit distribution such as beta, Kumaraswamy, and TL distributions have been used to generate the lifetime distribution in many researchers. Although there are researches tried to develop the generalized distribution with support on $(0,1)$ such as the T-Kumaraswamy family of distributions which has the members of family called generalized Kumaraswamy distributions introduced by Osatohanmwen *et al.* [18], the number of new unit distributions rarely find compared to the number of new lifetime distributions.

The aim of this work is to propose the T-Topp-Leone (T-TL) family of distributions and the obtained distributions called generalized TL distributions. The introduced distributions are not only used for fitting the proportional or percentage data but also for being the new generator in a parameter-adding method for generating statistical distributions in the future.

The rest of the paper is organized as follows. Under the materials and methods section, we define the T-TL family of distributions and study some associated properties. Furthermore, the generalized TL distributions are proposed. In the results and discussion section, three applications of the generalized TL distributions to real data sets are illustrated. Finally, the last section is conclusions.

Materials and methods

In this section, the T-TL family of distributions is proposed the probability distributions and some statistical properties. The obtained distributions called the generalized TL distributions will be introduced.

Genesis of family

Creating a new family of distributions requires 2 principal components, which are a generator and a parent distribution.

Definition 1 Let T be a random variable of a generator distribution with pdf $r(t)$ defined on $[a,b]$ for $-\infty < a < b < \infty$. Let X be a continuous random variable with cdf $F(x)$. Thus, the cdf of a new family of distributions is given by [12].

$$G(x) = \int_a^{W(F(x))} r(t)dt. \tag{1}$$

Let $W(F(x))$ be a function of $F(x)$ and satisfy the conditions following [12]:

1. $W(F(x)) \in [a,b]$,
2. $W(F(x))$ is differentiable and monotonically non-decreasing ,
3. $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$.

In 2014, Aljarrah *et al.* [13] proposed the T-$X\{Y\}$ family of distributions that took $W(F(x))$ in Eq. (1) to be the quantile function. Then, Alzaatreh *et al.* [14] presented the T-$R\{Y\}$ family which is re-defined from the T-$X\{Y\}$ family for convenience.

Definition 2 Let T be a random variable with cdf $F_T(x)$ and pdf $f_T(x)$ defined on $[a,b]$, for $-\infty < a < b < \infty$. Let R be a random variable with cdf $F_R(x)$ and pdf $f_R(x)$. Let Y be a random variable with cdf $F_Y(x)$, pdf $f_Y(x)$ defined on $[c,d]$, for $-\infty < c < d < \infty$. Its corresponding quantile function $Q_Y(p)$ when $0 < p < 1$. Thus, the cdf of a new family of distributions is given by [14];

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\ dt = P\Big[T \le Q_Y\left(F_R(x)\right)\Big] = F_T\Big(Q_Y\left(F_R(x)\right)\Big). \tag{2}$$

Its corresponding pdf is;

$$f_X(x) = f_R(x) \times Q_Y^{'}(F_R(x)) \times f_T\Big(Q_Y\left(F_R(x)\right)\Big) = f_R(x)\frac{f_T\Big(Q_Y\left(F_R(x)\right)\Big)}{f_Y\Big(Q_Y\left(F_R(x)\right)\Big)}, \tag{3}$$

where, $Q_Y^{'}(F_R(x)) = \dfrac{\mathrm{d}}{\mathrm{d}F_R(x)} Q_Y(F_R(x))$. Using $Q_Y(F_Y(x)) = x$, we then obtained $Q_Y^{'}(F_Y(x)) \times f_Y(x) = 1$, consequently $Q_Y^{'}(p) = 1/f_Y(Q_Y(p))$ and $p = F_R(x)$.

The T-TL family of distributions

Let X be a continuous random variable with cdf $F_X(x)$ and pdf $f_X(x)$. Using Eqs. (2) - (3) and $F_R(x) = x^\alpha(2-x)^\alpha$, the T-TL family of distributions is defined as;

$$F_X(x) = \int_0^{Q_Y(x^\alpha(2-x)^\alpha)} f_T(t)\ dt = F_T\Big(Q_Y(x^\alpha(2-x)^\alpha)\Big) \tag{4}$$

where, $0 < x < 1$ and $\alpha > 0$. Its corresponding pdf is;

$$f_X(x) \quad = \quad 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1} \frac{f_T\Big(Q_Y(x^\alpha(2-x)^\alpha)\Big)}{f_Y\Big(Q_Y(x^\alpha(2-x)^\alpha)\Big)}. \tag{5}$$

Remark 1 If X follows the T-TL family of distributions then

(i) $X \overset{\mathrm{d}}{\to} 1 - \left(1 - F_Y(T)^{1/\alpha}\right)^{1/2}$,

(ii) $Q_X(p) = 1 - \left(1 - F_Y(Q_T(p))^{1/\alpha}\right)^{1/2}$,

(iii) If $T \overset{\mathrm{d}}{\to} Y$ then $X \overset{\mathrm{d}}{\to}$ TL distribution with parameter α,

(iv) If $T \overset{\mathrm{d}}{\to}$ TL distribution with parameter α, then $X \overset{\mathrm{d}}{\to} T$.

The T-TL{exponential} distribution

If Y be a standard exponential random variable with pdf $f_Y(y) = \exp(-y)$ and its quantile function $Q_Y(p) = -\log(1-p)$, then $Q_Y(x^\alpha(2-x)^\alpha) = -\log\left(1 - x^\alpha(2-x)^\alpha\right)$. By using Eq. (4), the cdf of the T-TL{exponential} distribution is given by;

$$F_X(x) \quad = \quad F_T\Big(-\log\left(1 - x^\alpha(2-x)^\alpha\right)\Big). \tag{6}$$

From Eq. (5), its corresponding pdf can be written as;

$$f_X(x) \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1 - x^\alpha(2-x)^\alpha} f_T\Big(-\log\left(1 - x^\alpha(2-x)^\alpha\right)\Big), \tag{7}$$

where, $0 < x < 1$ and $\alpha > 0$.

The T-TL{extreme value} distribution

If Y be a standard extreme value random variable with $f_Y(y) = \exp\left[y - \exp(y)\right]$ and its quantile function $Q_Y(p) = \log(-\log(1-p))$, then $Q_Y(x^\alpha(2-x)^\alpha) = \log(-\log(1 - x^\alpha(2-x)^\alpha))$. By using Eq. (4), the cdf of the T-TL{extreme value} distribution is given by;

$$F_X(x) \quad = \quad F_T\bigg(\log\Big(-\log\left(1 - x^\alpha(2-x)^\alpha\right)\Big)\bigg). \tag{8}$$

From Eq. (5), its corresponding pdf can be written as;

$$f_X(x) = \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\left(x^\alpha(2-x)^\alpha - 1\right)\log\left(1 - x^\alpha(2-x)^\alpha\right)} f_T\bigg(\log\Big(-\log\left(1 - x^\alpha(2-x)^\alpha\right)\Big)\bigg), \tag{9}$$

where, $0 < x < 1$ and $\alpha > 0$.

The T-TL{log - logistic} distribution

If Y be a log-logistic random variable with pdf $f_Y(y) = (1+y)^{-2}$ and its quantile function $Q_Y(p) = \dfrac{p}{1-p}$, then $Q_Y(x^\alpha(2-x)^\alpha) = \dfrac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}$. By using Eq. (4), the cdf of the T-TL{log - logistic} distribution is given by;

$$F_X(x) \quad = \quad F_T\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right). \tag{10}$$

From Eq. (5), its corresponding pdf can be written as;

$$f_X(x) \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\left(1-x^\alpha(2-x)^\alpha\right)^2} f_T\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right), \tag{11}$$

where, $0 < x < 1$ and $\alpha > 0$.

The T-TL{logistic} distribution

If Y be a logistic random variable with pdf $f_Y(y) = \dfrac{\exp(-y)}{[1+\exp(-y)]^2}$ and its quantile function $Q_Y(p) = \log\left(\dfrac{p}{1-p}\right)$, then $Q_Y(x^\alpha(2-x)^\alpha) = \log\left(\dfrac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)$. By using Eq. (4), the cdf of the T-TL{logistic} distribution is given by;

$$F_X(x) \quad = \quad F_T\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right). \tag{12}$$

From Eq. (5), its corresponding pdf can be written as;

$$f_X(x) \quad = \quad \frac{2\alpha(1-x)}{x(2-x)\left(1-x^\alpha(2-x)^\alpha\right)} f_T\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right), \tag{13}$$

where, $0 < x < 1$ and $\alpha > 0$.

Statistical properties

We discuss some statistical properties of the T-TL family of distributions based on different quantile functions. They are including reliability and hazard functions, quantile function of T-TL family, Shannon entropy, moments, mean deviation and median deviation.

Reliability and hazard functions

If X follows the T-TL family of distributions with parameter $\alpha > 0$, the corresponding, reliability function is;

$$R_X(x) = 1 - F_T\Big(Q_Y\left(x^\alpha(2-x)^\alpha\right)\Big), \tag{14}$$

where, $0 < x < 1$ and $\alpha > 0$.

The reliability functions for the T-TL{exponential}, the T-TL{extreme value}, the T-TL{log-logistic} and the T-TL{logistic} distributions, are given respectively as;

$$R_X(x) = 1 - F_T\Big(-\log\big(1-x^\alpha(2-x)^\alpha\big)\Big), \tag{15}$$

$$R_X(x) = 1 - F_T\bigg(\log\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)\bigg), \tag{16}$$

$$R_X(x) = 1 - F_T\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right), \tag{17}$$

$$R_X(x) = 1 - F_T\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right). \tag{18}$$

Proof: For T-TL{exponential} distributions, the Eq. (15) is obtained by replacing $F_T\Big(Q_Y\left(x^\alpha(2-x)^\alpha\right)\Big)$ of Eqs. (14) to (6). The results of Eqs. (16) - (18) can be obtained by applying the same techniques.

Its corresponding hazard function is;

$$h_X(x) \quad = \quad \frac{f_X(x)}{R_X(x)} \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1-F_T\Big(Q_Y\left(x^\alpha(2-x)^\alpha\right)\Big)} \times \frac{f_T\Big(Q_Y\left(x^\alpha(2-x)^\alpha\right)\Big)}{f_Y\Big(Q_Y\left(x^\alpha(2-x)^\alpha\right)\Big)}. \tag{19}$$

The hazard functions of the random variable X for the T-TL family of distributions can be written as following;

(i) The T-TL{exponential} distribution;

$$h_X(x) \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1-x^\alpha(2-x)^\alpha} \times \frac{f_T\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)}{1-F_T\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)}, \tag{20}$$

(ii) The T-TL{extreme value} distribution;

$$h_X(x) \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\big(x^\alpha(2-x)^\alpha-1\big)\log\left(1-x^\alpha(2-x)^\alpha\right)} \times \frac{f_T\Big(\log\left(-\log(1-x^\alpha(2-x)^\alpha)\right)\Big)}{1-F_T\Big(\log\left(-\log(1-x^\alpha(2-x)^\alpha)\right)\Big)}, \tag{21}$$

(iii) The T-TL{log-logistic} distribution;

$$h_X(x) \quad = \quad \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\left(1-x^\alpha(2-x)^\alpha\right)^2} \times \frac{f_T\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)}{1-F_T\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)}, \tag{22}$$

(iv) The T-TL{logistic} distribution;

$$h_X(x) \quad = \quad \frac{2\alpha(1-x)}{x(2-x)\big(1-x^\alpha(2-x)^\alpha\big)} \times \frac{f_T\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right)}{1-F_T\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right)}. \tag{23}$$

Proof: The solutions of Eqs. (20) - (23) are obtained from replacing Eqs. (15) - (18) to $R_X(x)$ of Eq. (19) and then taking Eqs. (7), (9), (11) and (13) into$f_X(x)$ of Eq. (19), respectively.

Quantile function

Lemma 1 For any random variable T with density $f_T(x)$, then random variable;

(i) $X = 1-\sqrt{1-\big(1-\exp(-T)\big)^{1/\alpha}}$ follows the T-TL{exponential} distribution.

(ii) $X = 1-\sqrt{1-\Big(1-\exp\left(-\exp(T)\right)\Big)^{1/\alpha}}$ follows the T-TL{extreme value} distribution.

(iii) $X = 1 - \sqrt{1 - \left(\dfrac{T}{1+T}\right)^{1/\alpha}}$ follows the T-TL{log-logistic} distribution.

(iv) $X = 1 - \sqrt{1 - \left(\dfrac{\exp(T)}{1+\exp(T)}\right)^{1/\alpha}}$ follows the T-TL{logistic} distribution.

Proof: The proof is obtained from Remark 1(i). The **Lemma 1** gives the relationships between the random variable X and the random variable T. The results show that the random samples from X can be generated by using random variable T.

Lemma 2 If T and X be a random variables with corresponding quantile functions $Q_T(p)$ and $Q_X(p)$ when $0 < p < 1$ then the quantile functions for the T-TL{exponential}, the T-TL{extreme value}, the T-TL{log-logistic} and the T-TL{logistic} distributions, are given, respectively as;

(i) $Q_X(p) = 1 - \sqrt{1 - \Big(1 - \exp\big(-Q_T(p)\big)\Big)^{1/\alpha}}$,

(ii) $Q_X(p) = 1 - \sqrt{1 - \Big(1 - \exp\big(-\exp\big(Q_T(p)\big)\big)\Big)^{1/\alpha}}$,

(iii) $Q_X(p) = 1 - \sqrt{1 - \left(\dfrac{Q_T(p)}{1+Q_T(p)}\right)^{1/\alpha}}$,

(iv) $Q_X(p) = 1 - \sqrt{1 - \left(\dfrac{\exp(Q_T(p))}{1+\exp(Q_T(p))}\right)^{1/\alpha}}$.

Proof: The proof follows from Remark 1(ii).

Shannon entropy

If X be a random variable with pdf $f(x)$ then $\mathbb{E}\big[-\log(f(x))\big]$ is defined by Shannon [19]. It is a measure of variation of uncertainty [20].

Theorem 1 The Shannon entropy of the T-TL family of distributions is given by;

$$\begin{aligned} \eta_X &= -\log 2 - \log\alpha - (\alpha-1)\mathbb{E}[\log x] - \mathbb{E}[\log(1-x)] - (\alpha-1)\mathbb{E}[\log(2-x)] + \eta_T \\ &\quad +\mathbb{E}[\log f_Y(T)], \end{aligned} \tag{24}$$

where η_T is the Shannon entropy of the distribution of the random variable T.

Proof: From Remark 1(i), it follows that $T = Q_Y(x^\alpha(2-x)^\alpha)$. Taking the expectation of the negative logarithm of the pdf in Eq. (5) gives the required result in Eq. (24).

Corollary 1 The Shannon entropy for the T-TL family of distributions are written as:

(i) The T-TL{exponential} distribution:

$$\eta_X = -\log 2 - \log\alpha - (\alpha-1)\mathbb{E}[\log x] - \mathbb{E}[\log(1-x)] - (\alpha-1)\mathbb{E}[\log(2-x)] + \eta_T - \mu_T,$$

(ii) The T-TL{extreme value} distribution:

$$\begin{aligned} \eta_X &= -\log 2 - \log\alpha - (\alpha-1)\mathbb{E}[\log x] - \mathbb{E}[\log(1-x)] - (\alpha-1)\mathbb{E}[\log(2-x)] + \eta_T + \mu_T \\ &\quad -\mathbb{E}\left[\exp(T)\right], \end{aligned}$$

(iii) The T-TL{log-logistic} distribution:

$$\begin{aligned}\eta_X &= -\log 2-\log\alpha-(\alpha-1)\mathbb{E}[\log x]-\mathbb{E}[\log(1-x)]-(\alpha-1)\mathbb{E}[\log(2-x)]+\eta_T\\ &\quad -2\mathbb{E}[\log(1+T)],\end{aligned}$$

(iv) The T-TL{logistic} distribution:

$$\begin{aligned}\eta_X &= -\log 2-\log\alpha-(\alpha-1)\mathbb{E}[\log x]-\mathbb{E}[\log(1-x)]-(\alpha-1)\mathbb{E}[\log(2-x)]+\eta_T-\mu_T\\ &\quad -2\mathbb{E}\left[\log(1+\exp(-T))\right],\end{aligned}$$

where μ_T is the mean of random variable T. The results in Corollary 1 are using Eq. (24) and $f_Y(T)=\exp(-T),\ \exp\left(T-\exp(T)\right),\ (1+T)^{-2},\ \text{and}\ \exp(-T)\left(1+\exp(-T)\right)^2$ for the exponential, extreme value, log-logistic and logistic distributions, respectively.

Moments

Theorem 2 If X be a random variable, the r^{th} non-central moments of the T-TL{exponential}, the T-TL{extreme value}, the T-TL{log-logistic} and the T-TL{logistic} distributions, are given, respectively as;

$$\text{(i)}\quad \mathbb{E}\left[X^r\right]=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\binom{r}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}\mathbb{E}[T^{k_4}]\ , \tag{25}$$

$$\text{(ii)}\quad \mathbb{E}\left[X^r\right]=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty}\binom{r}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}k_4^{k_5}}{k_4!k_5!}\mathbb{E}[T^{k_5}], \tag{26}$$

$$\text{(iii)}\quad \mathbb{E}\left[X^r\right]=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{r}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}\mathbb{E}\left[\left(\frac{T}{1+T}\right)^{k_2/\alpha}\right], \tag{27}$$

$$\text{(iv)}\quad \mathbb{E}\left[X^r\right]=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{r}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}\mathbb{E}\left[\left(\frac{\exp(T)}{1+\exp(T)}\right)^{k_2/\alpha}\right]. \tag{28}$$

Proof: By using Lemma 1, we prove Eq. (25) to be an example. The r^{th} non-central moments of the T-TL{exponential} distribution can be given as $\mathbb{E}(X^r)=\mathbb{E}\left[1-\sqrt{1-(1-\exp(-T))^{1/\alpha}}\right]$. Then, using the binomial expansion formula and taking the expectation solve solution Eq. (25). The results of Eqs. (26) - (28) can be obtained by applying the same techniques for Eq. (25).

Mean deviation and median deviation

The mean deviation and the median deviation are defined as $D(\mu)$ and $D(M)$, respectively. These are used to measure the dispersion and the spread in a population from the center.

Theorem 3 The $D(\mu)$ and $D(M)$ for the T-TL family of distributions are given as;

(i) The T-TL{exponential} distribution:

$$D(\mu)=2\mu F_X(\mu)-2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}S_u(\mu,0,k_4), \tag{29}$$

$$D(M)=\mu-2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}S_u(M,0,k_4), \tag{30}$$

(ii) The T-TL{extreme value} distribution:

$$D(\mu) = 2\mu F_X(\mu) - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}k_4^{k_5}}{k_4!k_5!} \times S_u(\mu, -\infty, k_5), \tag{31}$$

$$D(M) = \mu - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}k_4^{k_5}}{k_4!k_5!} \times S_u(M, -\infty, k_5), \tag{32}$$

(iii) The T-TL{log-logistic} distribution:

$$D(\mu) = 2\mu F_X(\mu) - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}S_{\frac{u}{1+u}}(\mu, 0, k_2/\alpha), \tag{33}$$

$$D(M) = \mu - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}S_{\frac{u}{1+u}}(M, 0, k_2/\alpha), \tag{34}$$

(iv) The T-TL{logistic} distribution:

$$D(\mu) = 2\mu F_X(\mu) - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}S_{\frac{e^u}{1+e^u}}(\mu, -\infty, k_2/\alpha), \tag{35}$$

$$D(M) = \mu - 2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}(-1)^{k_1+k_2}S_{\frac{e^u}{1+e^u}}(M, -\infty, k_2/\alpha), \tag{36}$$

where $S_z(c, a, n) = \int_a^{Q_Y(c^\alpha(2-c)^\alpha)} z^n f_T(u)\mathrm{d}u.$

Proof: By definition of $D(\mu)$ and $D(M)$, we have

$$\begin{aligned} D(\mu) &= \int_0^\mu (\mu - x)f_X(x)\mathrm{d}x + \int_\mu^1 (x-\mu)f_X(x)\mathrm{d}x = 2\int_0^\mu (\mu - x)f_X(x)\mathrm{d}x \\ &= 2\mu F_X(\mu) - 2\int_0^\mu x f_X(x)\mathrm{d}x. \end{aligned} \tag{37}$$

$$\begin{aligned} D(M) &= \int_0^M (M-x)f_X(x)\mathrm{d}x + \int_M^1 (x-M)f_X(x)\mathrm{d}x \\ &= 2\int_0^M (M-x)f_X(x)\mathrm{d}x + \mu - M = \mu - 2\int_0^M x f_X(x)\mathrm{d}x. \end{aligned} \tag{38}$$

We prove Eq. (29) for the T-TL{exponential} distribution. Defining the integral

$$I_c = \int_0^c x f_X(x)\mathrm{d}x = 2\alpha\int_0^c \frac{x^\alpha(1-x)(2-x)^{\alpha-1}}{1-x^\alpha(2-x)^\alpha} f_T\big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\big)\mathrm{d}x, \tag{39}$$

and using the substitution $u = -\log\left(1 - x^\alpha(2-x)^\alpha\right)$ can be written as;

$$I_c = \int_0^{-\log\left(1-x^\alpha(2-x)^\alpha\right)} 1 - \sqrt{1-(1-e^{-u})^{1/\alpha}}\, f_T(u)\ \mathrm{d}u. \tag{40}$$

Using the result of the generalized binomial expansion in Theorem 2, Eq. (40) can be given as;

$$I_c = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}S_u(c,0,k_4), \tag{41}$$

where $S_z(c,a,n) = \int_a^{Q_Y(c^\alpha(2-c)^\alpha)} z^n f_T(u)\mathrm{d}u$ and $Q_Y(x^\alpha(2-x)^\alpha) = -\log\left(1-x^\alpha(2-x)^\alpha\right)$.

The results in Eqs. (29) - (30) follow by taking Eq. (41) in Eqs. (37) - (38). Applying the same techniques of showing Eqs. (29) - (30), the results of Eqs. (31) - (32) for (ii), Eqs. (33) - (34) for (iii), Eqs. (35) - (36) for (iv).

Some examples of the T-TL family of distributions

We introduce some distributions of the T-TL family of distributions called generalizations TL distributions that are generated from different T distributions according to the standard Y distributions. These generalizations include the Weibull-TL{exponential}, the log-logistic-TL{exponential}, the logistic-TL{extreme value}, exponential-TL{log-logistic} and the normal-TL{logistic} distributions. Some properties of Weibull-TL{exponential} are given. To conserve space, the properties of the other distributions are not provided but we can study with the same method.

The Weibull-TL{exponential} (WTL) distribution

If a random variable T follows the Weibull distribution with parameters λ and θ, which has the cdf $F_T(t) = 1-\exp\left(-\lambda t^\theta\right)$ and the pdf $f_T(t) = \lambda\theta t^{\theta-1}\exp\left(-\lambda t^\theta\right)$. According to Eq. (6), the cdf of the WTL distribution is defined as

$$F_X(x) = 1-\exp\left[-\lambda\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)^\theta\right]. \tag{42}$$

By using Eq. (7), its corresponding pdf is given by;

$$\begin{aligned} f_X(x) &= \frac{2\alpha\lambda\theta x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)^{\theta-1}}{1-x^\alpha(2-x)^\alpha} \\ &\quad\times\exp\left[-\lambda\Big(-\log\left(1-x^\alpha(2-x)^\alpha\right)\Big)^\theta\right], \end{aligned} \tag{43}$$

where $0<x<1, \alpha>0, \lambda>0,$ and $\theta>0$. If $\theta=1$, the WTL distribution reduces to the exponential-TL{exponential} distribution and if $\lambda=\theta=1$, the WTL distribution reduces to the TL distribution.

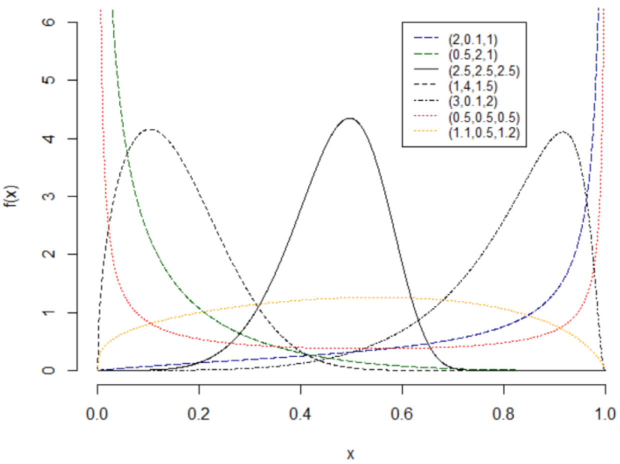

Figure 1 Density plots of the WTL distribution for specified parameters of (α,λ,θ).

Some mathematical properties of WTL are provided in the following by using the general properties discussed in statistical properties. Let X be WTL (α,λ,θ) distribution, some mathematical properties are

1) Reliability function:

$$R_X(x)=\exp\Big(-\lambda\Big(-\log\big(1-x^{\alpha}(2-x)^{\alpha}\big)\Big)^{\theta}\Big).$$

2) Hazard function:

$$h_X(x) \;=\; \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{1-x^{\alpha}(2-x)^{\alpha}}\times\;\lambda\theta\Big(-\log\big(1-x^{\alpha}(2-x)^{\alpha}\big)\Big)^{\theta-1}.$$

3) Quantile function:

$$Q_X(p)=1-\sqrt{1-\Big(1-\exp\big(-(-\lambda^{-1}\log(1-p))^{1/\theta}\big)\Big)^{1/\alpha}}.$$

4) Shannon entropy: By using Corollary 1 and given that $\mu_T=\frac{1}{\lambda^{1/\theta}}\Gamma(1+1/\theta)$, and $\eta_T=1+\gamma\Big(1-\frac{1}{\theta}\Big)+\log\Big(\frac{1}{\theta\lambda^{1/\theta}}\Big)$ [21] the Shannon entropy of the WTL distribution is given by;

$$\begin{aligned}\eta_X \;&=\; -\log 2-\log\alpha-(\alpha-1)\mathbb{E}[\log x]-\mathbb{E}[\log(1-x)]-(\alpha-1)\mathbb{E}[\log(2-x)]\\ &\quad+1+\gamma\Big(1-\frac{1}{\theta}\Big)+\log\Big(\frac{1}{\theta\lambda^{1/\theta}}\Big)-\frac{1}{\lambda^{1/\theta}}\Gamma(1+1/\theta),\end{aligned}$$

where γ is the Euler's constant and $\Gamma(n)=(n-1)!$ is the complete gamma function.

5) Moments: By using **Theorem 2** and $\mathbb{E}\left[T^{k_4}\right]=\Big(\frac{1}{\lambda^{k_4/\theta}}\Big)\Gamma(1+k_4/\theta)$, the r^{th} non-central moments of the WTL distribution is defined as;

$$\mathbb{E}\left[X^r\right]=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\binom{r}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}\Big(\frac{1}{\lambda^{k_4/\theta}}\Big)\Gamma(1+k_4/\theta).$$

6) Mean deviation: Following **Theorem 3** and replacing $S_u(\mu,0,k_4),$ and $S_u(M,0,k_4)$ with $\Big(\frac{1}{\lambda^{k_4/\theta}}\Big)\Gamma\Big(1+k_4/\theta,\lambda\Big(-\log\big(1-x^{\alpha}(2-x)^{\alpha}\big)\Big)^{\theta}\Big)$, the mean deviation from the mean and mean deviation from the median of WTL are given, respectively by;

$$\begin{aligned}D(\mu)=2\mu F_X(\mu)-2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}&\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}\\ &\times\Big(\frac{1}{\lambda^{k_4/\theta}}\Big)\Gamma\Big(1+k_4/\theta,\lambda\Big(-\log\big(1-\mu^{\alpha}(2-\mu)^{\alpha}\big)\Big)^{\theta}\Big),\end{aligned}$$

$$\begin{aligned}D(M)=\mu-2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}&\binom{1}{k_1}\binom{k_1/2}{k_2}\binom{k_2/\alpha}{k_3}\frac{(-1)^{\sum_{i=1}^{4}k_i}k_3^{k_4}}{k_4!}\\ &\times\Big(\frac{1}{\lambda^{k_4/\theta}}\Big)\Gamma\Big(1+k_4/\theta,\lambda\Big(-\log\big(1-M^{\alpha}(2-M)^{\alpha}\big)\Big)^{\theta}\Big),\end{aligned}$$

where $\Gamma(m,x)=\int_0^x u^{m-1}e^{-u}\mathrm{d}u$ is the incomplete gamma function.

The log-logistic-TL {exponential} (LLTL) distribution

If a random variable T follows the log-logistic distribution with parametersν and β, which has the cdf $F_T(t) = \left[1+(t/\nu)^{-\beta}\right]^{-1}$and pdf $f_T(t) = \frac{(\beta/\nu)(t/\nu)^{\beta-1}}{\left[1+(t/\nu)^{\beta}\right]^2}$. According to Eq. (6), the cdf of the LLTL distribution is defined as;

$$F_X(x) \quad = \quad \left[1+\left(-\nu^{-1}\log\left(1-x^{\alpha}(2-x)^{\alpha}\right)\right)^{-\beta}\right]^{-1}. \tag{44}$$

By using Eq. (7), its corresponding pdf is given by;

$$f_X(x) = \frac{2\alpha\beta x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\nu\left(1-x^{\alpha}(2-x)^{\alpha}\right)} \frac{\left[-\nu^{-1}\log\left(1-x^{\alpha}(2-x)^{\alpha}\right)\right]^{\beta-1}}{\left[1+\left(-\nu^{-1}\log\left(1-x^{\alpha}(2-x)^{\alpha}\right)\right)^{\beta}\right]^2}, \tag{45}$$

where $0 < x < 1, \alpha > 0, \nu > 0,$ and $\beta > 0$.

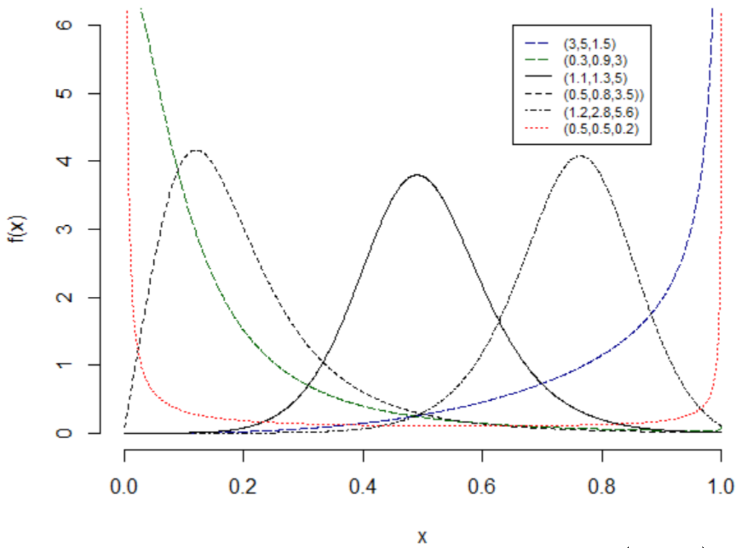

Figure 2 Density plots of the LLTL distribution for specified parameters of (α, ν, β).

The logistic-TL{extreme value} (LTL) distribution

If a random variable T follows the logistic distribution with parameters μ and δ, which has the cdf $F_T(t) = \left(1+\exp\left(-\frac{t-\mu}{\delta}\right)\right)^{-1}$ and the pdf $f_T(t) = \frac{\exp\left(-\frac{t-\mu}{\delta}\right)}{\delta\left(1+\exp\left(-\frac{t-\mu}{\delta}\right)\right)^2}$. According to Eq. (8), the cdf of the LTL distribution is defined as;

$$F_X(x) \quad = \quad \left(1+\exp\left(-\frac{\log(-\log\left(1-x^{\alpha}(2-x)^{\alpha}\right))-\mu}{\delta}\right)\right)^{-1}. \tag{46}$$

By using Eq. (9), its corresponding pdf is given by;

$$f_X(x) = \frac{2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\delta\Big(x^{\alpha}(2-x)^{\alpha}-1\Big)\log\Big(1-x^{\alpha}(2-x)^{\alpha}\Big)} \times \frac{\exp\Big[\delta^{-1}\Big(\mu-\log\big(-\log\big(1-x^{\alpha}(2-x)^{\alpha}\big)\big)\Big)\Big]}{\Big[1+\exp\Big[\delta^{-1}\Big(\mu-\log\big(-\log\big(1-x^{\alpha}(2-x)^{\alpha}\big)\big)\Big)\Big]\Big]^2}, \tag{47}$$

where $0 < x < 1,\ \alpha > 0, -\infty < \mu < \infty,$ and $\delta > 0$. If $\mu = 0$, the LTL distribution reduces to the LLTL distribution with parameters $\alpha,\ \nu = 1,$ and $\beta = 1/\delta$.

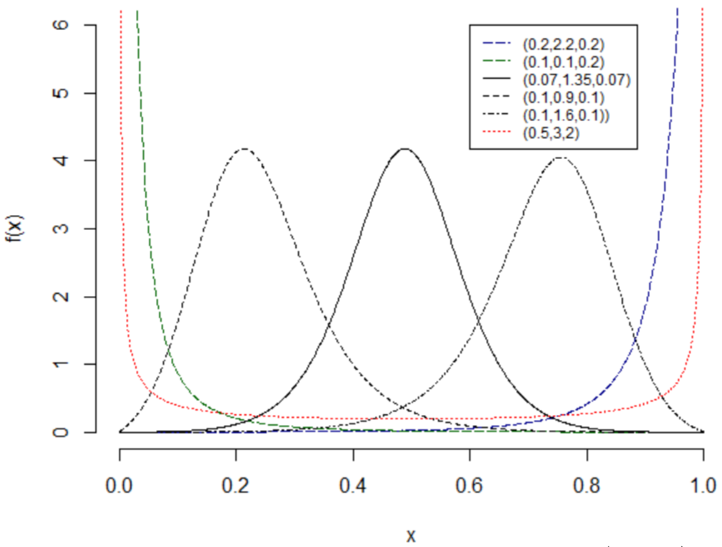

Figure 3 Density plots of the LTL distribution for specified parameters of (α, μ, δ).

The exponential-TL{log-logistic} (ETL) distribution

If a random variable T follows the exponential distribution with parameter λ, which has the cdf $F_T(t) = 1 - \exp(-\lambda t)$ and the pdf $f_T(t) = \lambda \exp(-\lambda t)$. According to Eq. (10), the cdf of the ETL distribution is defined as;

$$F_X(x) = 1 - \exp\left[-\lambda\left(\frac{x^{\alpha}(2-x)^{\alpha}}{1-x^{\alpha}(2-x)^{\alpha}}\right)\right]. \tag{48}$$

By using Eq. (11), its corresponding pdf is given by;

$$f_X(x) = \frac{2\alpha\lambda x^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{\Big(1-x^{\alpha}(2-x)^{\alpha}\Big)^2}\exp\left[-\lambda\left(\frac{x^{\alpha}(2-x)^{\alpha}}{1-x^{\alpha}(2-x)^{\alpha}}\right)\right], \tag{49}$$

where $0 < x < 1, \alpha > 0,$ and $\lambda > 0$.

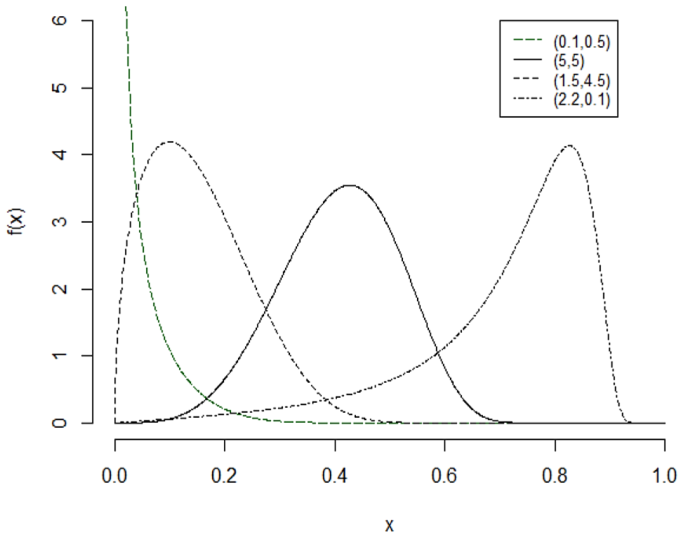

Figure 4 Density plots of the ETL distribution for specified parameters of (α, λ).

The normal-TL{logistic} (NTL) distribution

A random variable T is said to follow the normal distribution with parameters μ and σ, which has the cdf $F_T(t) = \Phi(t)$ and the pdf $f_T(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]$. According to Eq. (12), the cdf of the NTL distribution is defined as

$$F_X(x) = \Phi\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)\right). \tag{50}$$

By using Eq. (13), its corresponding pdf is given by;

$$f_X(x) = \frac{2\alpha(1-x)}{x\sqrt{2\pi\sigma^2}(2-x)\left(1-x^\alpha(2-x)^\alpha\right)} \exp\left[-\frac{1}{2\sigma^2}\left(\log\left(\frac{x^\alpha(2-x)^\alpha}{1-x^\alpha(2-x)^\alpha}\right)-\mu\right)^2\right], \tag{51}$$

where $0 < x < 1, \alpha > 0, -\infty < \mu < \infty,$ and $\sigma > 0$.

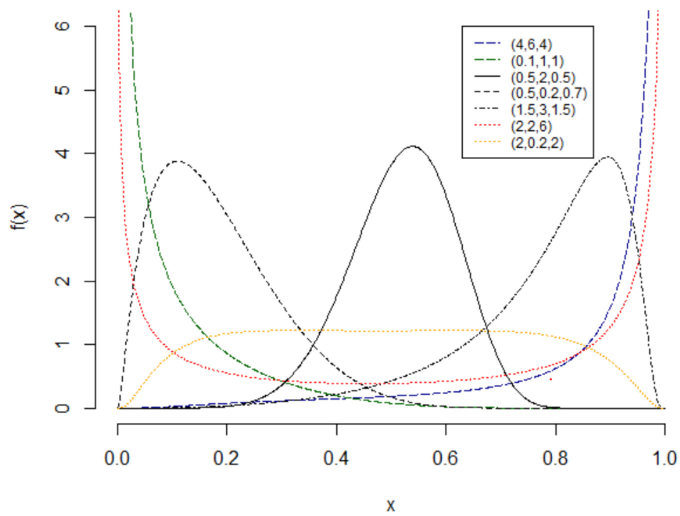

Figure 5 Density plots of the NTL distribution for specified parameters of (α, μ, σ).

Results and discussion

In this section, we present flexibility of the proposed generalized TL distributions. We compare the generalized TL distributions including the WTL, LLTL, LTL, ETL and NTL distributions with other distributions, such as TL, beta (BETA) and Kumaraswamy (KUM) distributions, and estimate parameters by maximum likelihood estimation (MLE) . The MLE of the distribution parameters are computed by maximizing the log-likelihood function $LL = \sum_{i=1}^{n} log(f(x_i))$ with respect to each parameter. Then, we obtain estimated parameters by solving the system of non-linear equations numerically through a numerical method by using `optim` function from `stats` package in R programming language [22]. The criteria for choosing the best fitted distribution in application study will be the Kolmogorov-Smirnov (KS) test. The formula for the KS test can be given as $D = \sup_x |F^*(x) - S(x)|$, where $F^*(x)$ is the cdf of the hypothesized distribution and $S(x)$ is the empirical distribution function of a sample. We used `ks.test` function from `dgof` package [23] in R programming language to calculate this statistic. The results which contain the parameter estimates, the log-likelihood (LL) values, the values of the KS test statistic and its *p*-value for all the distributions are shown in table. Moreover the figures show the empirical and fitted distributions of the data sets. We consider the highest log-likelihood or the smallest values of KS that give the best fit for the data. Three real data sets are given below:

Data I: The first data set represents the hole diameter is 12 mm, and the sheet thickness is 3.15 mm. This data set consists of 50 observations (in the unit of millimeter) obtained from Dasgupta [24]. The hole diameter readings are taken on work and related with the machine. The data are as follows: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18 and 0.16.

Data II: The second data consists of the first 58 observations of the failure time of Kevlar 49/epoxy strands test at 90 % stress level. This data set is obtained from Andrews and Herzberg [25]. The data are as follows: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95 and 0.99.

Data III: The third data set refers to the shape perimeter by squared (area) from measurements on petroleum rock samples obtained from Cordeiro and Brito [26]. The 48 rock samples were collected from a petroleum reservoir. This data are as follows: 0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320 and 0.2004470.

Table 1 The parameter estimates and some statistics of model fitting to Data I.

Distributions	WTL	LLTL	LTL	ETL
Parameter Estimates	$\hat{\alpha}$ = 0.0995 $\hat{\lambda}$ = 0.0042 $\hat{\theta}$ = 6.5647	$\hat{\alpha}$ = 1.166e-08 $\hat{\nu}$ = 18.0538 $\hat{\beta}$ = 75.0089	$\hat{\alpha}$ = 10.6802 $\hat{\mu}$ = −13.8205 $\hat{\delta}$ = 3.5697	$\hat{\alpha}$ = 2.0453 $\hat{\lambda}$ = 8.3821
LL	57.1473	54.9198	48.3703	56.3481
KS test (*p*-value)	0.0905 (0.8077)	0.0990 (0.7113)	0.1416 (0.2690)	0.1151 (0.5213)
Distributions	**NTL**	**TL**	**BETA**	**KUM**
Parameter Estimates	$\hat{\alpha}$ = 96.5179 $\hat{\mu}$ = −133.1667 $\hat{\sigma}$ = 60.6739	$\hat{\alpha}$ = 0.7248	$\hat{\alpha}$ = 2.6827 $\hat{\beta}$ = 13.8659	$\hat{a}$ = 2.0774 $\hat{b}$ = 33.1356
LL	46.7693	28.4078	54.6067	56.0687
KS test (*p*-value)	0.2017 (0.0343)	0.3623 (< 0.0001)	0.1414 (0.2699)	0.1103 (0.5776)

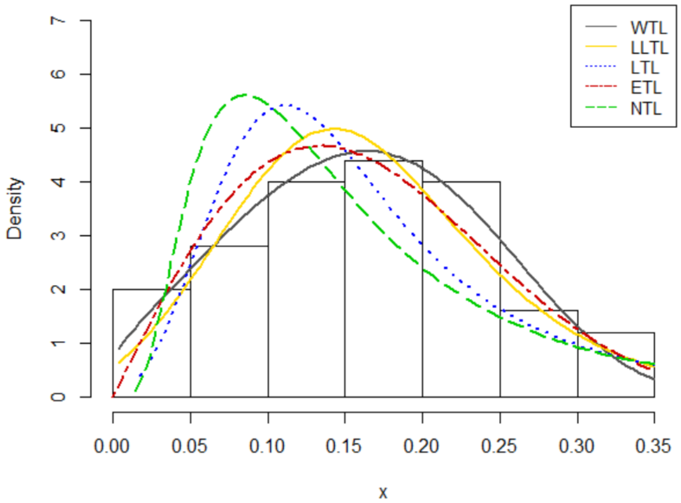

Figure 6 Empirical and fitted distributions for Data I.

The Data I is nearly symmetric as shown in **Figure 6**. All generalized TL distributions are better fit than TL distribution. In addition, the result from **Table 1** shows that the WTL and LLTL are more appropriate for fitting Data I than other distributions. The WTL distribution, especially, provides the smallest of KS statistic and has the highest *p*-value corresponding to distribution that the highest LL value.

Table 2 The parameter estimates and some statistics of model fitting to Data II.

Distributions	**WTL**	**LLTL**	**LTL**	**NTL**
Parameter Estimates	$\hat{\alpha}$ = 0.9688 $\hat{\lambda}$ = 0.8095 $\hat{\theta}$ = 0.7910	$\hat{\alpha}$ = 0.2440 $\hat{\nu}$ = 1.9539 $\hat{\beta}$ = 2.2073	$\hat{\alpha}$ = 0.2439 $\hat{\mu}$ = 0.6699 $\hat{\delta}$ = 0.4530	$\hat{\alpha}$ = 1.9643 $\hat{\mu}$ = - 0.9847 $\hat{\sigma}$ = 3.3476
LL	5.7848	3.9495	3.9495	8.1068
KS test (*p*-value)	0.0980 (0.6335)	0.0968 (0.6494)	0.0968 (0.6492)	0.0593 (0.9868)
Distributions	**TL**	**BETA**	**KUM**	
Parameter Estimates	$\hat{\alpha}$ = 0.9507	$\hat{\alpha}$= 0.6776 $\hat{\beta}$ = 1.0412	$\hat{a}$ = 0.6825 $\hat{b}$ = 1.0472	
LL	2.3420	5.6714	5.6824	
KS test (*p*-value)	0.1921 (0.0277)	0.1038 (0.5593)	0.1035 (0.5629)	

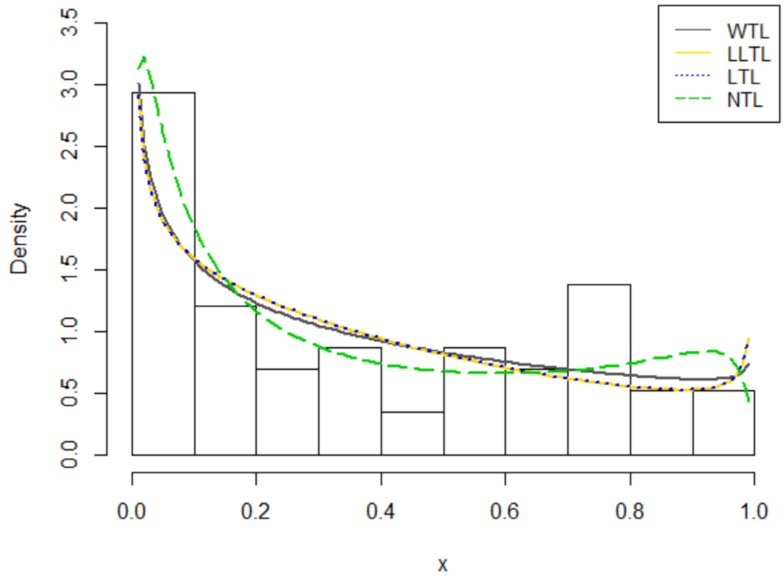

Figure 7 Empirical and fitted distributions for Data II.

The **Figure 7** has shown that the curve of Data II is decreasing. From **Table 2**, it can be observed that all the generalized TL distributions are better fit than the TL, BETA and KUM distributions reported by higher *p*-value of KS statistic. The ETL cannot evaluate appropriate parameters for this data. The NTL distribution, that provides the highest *p*-value of KS statistic and LL value, is the best fit for Data II.

Table 3 The parameter estimates and some statistics of model fitting to Data III.

Distributions	**WTL**	**LLTL**	**LTL**	**ETL**
Parameter Estimates	$\hat{\alpha}$ = 14.2653 $\hat{\lambda}$ = 16.3412 $\hat{\theta}$ = 0.2291	$\hat{\alpha}$ = 3.5930 $\hat{\nu}$ = 0.0262 $\hat{\beta}$ = 1.5395	$\hat{\alpha}$ = 3.5950 $\hat{\mu}$ = - 3.6431 $\hat{\delta}$ = 0.6499	$\hat{\alpha}$ = 3.0026 $\hat{\lambda}$ = 11.1555
LL	54.0540	57.6197	57.6197	52.3404
KS test (*p*-value)	0.14402 (0.2724)	0.0841 (0.8861)	0.0841 (0.8861)	0.1652 (0.1453)
Distributions	**NTL**	**TL**	**BETA**	**KUM**
Parameter Estimates	$\hat{\alpha}$ = 5.6079 $\hat{\mu}$ = -5.6529 $\hat{\sigma}$ = 1.7497	$\hat{\alpha}$ = 0.9894	$\hat{\alpha}$ = 5.9420 $\hat{\beta}$ = 21.2065	$\hat{a}$ = 2.7187 $\hat{b}$ = 44.6540
LL	58.1919	21.1660	55.6002	52.4915
KS test (*p*-value)	0.0965 (0.7623)	0.3680 (< 0.0001)	0.1428 (0.2820)	0.1533 (0.2091)

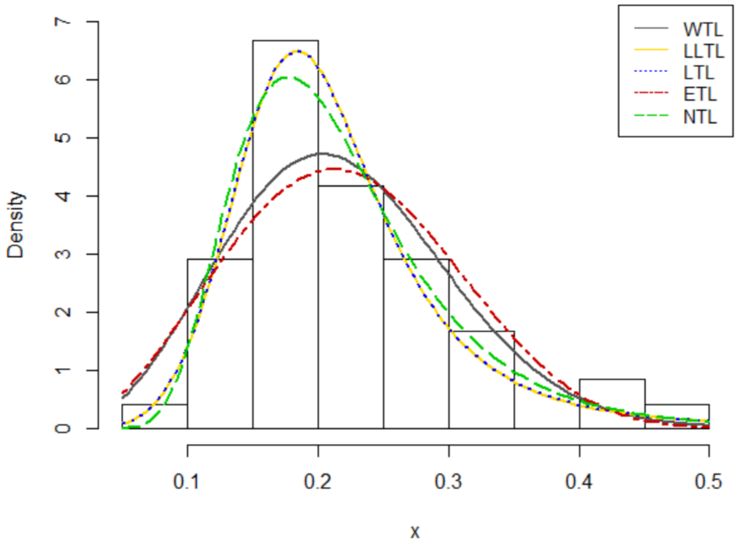

Figure 8 Empirical and fitted distributions for data III.

The Data III is right skew curve as shown in **Figure 8**. **Table 3** reports that all generalized TL distributions are more efficient than TL distribution. The LLTL and LTL distributions have the smallest KS statistics, highest *p*-value, and largest LL value, which indicate that the LLTL and LTL distributions are superior to the other distributions.

Conclusions

In this paper, the T-Topp-Leone family of distributions which provides distributions bounded on (0,1) called generalized Topp-Leone distributions including the Weibull-Topp-Leone{exponential}, the log-logistic-Topp-Leone{exponential}, the logistic-Topp-Leone{extreme-value}, the exponential-Topp-Leone{log-logistic} and the normal-Topp-Leone{logistic} distributions is proposed. Some statistical properties, such as reliability function, hazard function, quantile function of T-Topp-Leone family, Shannon entropy, moments, mean deviation and median deviation are discussed. All generalized Topp-Leone distributions are applied to 3 real datasets and the results indicated that five distributions obtained from the new family can be used as good alternatives to the Topp-Leone, beta and Kumaraswamy distributions. We hope that the proposed family of distributions will attract wider applications for the analysis of proportion and percentage data which we generally observe in various field of reliability, engineering, medicine, etc. Furthermore, we can provide alternative regression models for the response variable with support on (0,1).

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