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On the Solution of the Variable Order Time Fractional Schrödinger Equation

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Abstract

In the present paper, we used the Adomian decomposition method to obtain solutions of linear and nonlinear variable order fractional Schrödinger equations. Twelve illustrative applications have been presented. When the fractional order is unity, the results are the same as those obtained by the standard integer order derivative. Some of the obtained results are discussed graphically. Some of the obtained results 1st appear in the literature. This technique can be applied to other linear or nonlinear fractional differential models.

Keywords: Fractional partial differential equation, Variable order fractional Schrödinger equations, Adomian decomposition, Adomian polynomials, Caputo fractional derivative

Introduction

Nick Laskin's fractional quantum mechanics were developed through the proposed new path integrals over measures induced by the Levy flights stochastic process. Based on its suggestion of a new quantic mechanics fractional and statistical mechanics, he generalized the path integral component of Feynman and Wiener. Furthermore, Laskin used the quantum Riesz fractional derivative in deriving the space-fractional Schrödinger equation for the wave function of the quantum mechanical system [1].

The properties of the fractional Schrödinger equation (FSE) are investigated by Laskin [2]. The 1st attempt in obtaining the solution of FSE is given by Naber in 2004, that by solving the time FSE for a free particle and a potential well [3]. Guo *et al*. [4] investigated the existence and uniqueness of the global smooth solution to the period boundary value problem of the FSE equation via the energy method. Dong and Xu [5], studied the space-time FSE in sense of Caputo fractional derivative and the quantum Riesz fractional operator, and they expressed the time evolution operator of wave in terms of a Mittag-Leffler function. Rida *et al*. [6] used the Adomian decomposition method in finding analytical and approximate solutions to time-fractional nonlinear Schrödinger equation (tFNLSE) with Caputo fractional derivative under certain initial conditions. Eid *et al*. [7], solved space FSE with Coulomb potential. Abdel-Salam *et al*. [8], and Yousif *et al*. [9] studied the space-time FNLSE.

Coimbra [10] defined a variable order (VO) differential operator to describe a mathematical framework for solving complex mechanical problems; the VO operator is used to illustrate the dynamic behavior of VO frictional forces. Soon *et al*. [11] studied the complex dynamics of a variable viscoelasticity oscillator using the concept of VO-calculus; a numerical method is developed to describe the VO fractional differential equation (VOFDE) behavior with 2nd-order accuracy. Ramirez and Coimbra [12] proposed a VO constitutive relation model to describe the linear viscoelastic response, they considered the order of the derivative to be a function of time. Zhuang *et al*. [13] proposed explicit and implicit Euler approximations for a VOF advection-diffusion equation with a nonlinear source term on a finite domain.

Lin *et al*. [14] examined a nonlinear fractional diffusion equation of VO, which is considered in generalised Riesz sense as a fractional derivative of VO. In approximating a solution to the problem, they proposed an explicit finite differences scheme.

Abbas *et al*. [15] have studied the existence, using Caputo fractional derivatives, of solutions to partial differential equations with impulses at various times and with endless delays.

Sun *et al*. [16] initiated a comparative study into the memory properties of systems for the integrated order derivatives, the constant order fractional derivatives, and 2 types of VOF derivatives. It was found that the integer order derivative can be utilized to characterize short system memory, while the fractional constant order derivatives benefit from characterizing long system memory, whereas the VOF derivative can be used for displaying variable system memory.

The solution to a generalised, fractional equation with non-self-supported VOs was discussed by Razminia *et al*. [17]. In order to study the VOF wave equations, Sweilam *et al*. [18] have employed an explicit finite difference. In the sense that Riesz describes the fractional derivative. The stability and convergence of the proposed method were analyzed.

Zhang [19] introduced the nonlinear boundary value VO differential equation solution. Their analysis and Arzela-Ascoli theorem discussed the existence of the solution. The existence and unique outcomes of VOFDEs were studied by Xu and He [20]. The existence of VOFDEs as the result of the Cauchy issue is obtained through the construction of the analytical solution iteration series. The result of uniqueness is achieved through the principle of contract mapping.

In order to obtain the numerical FDE solution with a variable coefficient, Chen *et al*. [21] have developed a general formula for the VO Legendre general function. The implicit finite difference method was introduced in non-homo General porous media by Chen *et al*. [22] for the bi-dimensional VOF percolation equation and the stability and convergence of the approach. Zeng *et al*. [23] generalized existing Jacobi-Gauss-Lobatto collocation methods for VOFDEs with the singular weighted Jacobi polynomial approximation functions. The proposed spectral collocation method was used to solve partial and ordinary differential equations with the specific features of the endpoint.

In order to generalize the time-Fractional telegraph equation, Atangana [24] used the concept of a derivative of the Fractional VO; solved the problem numerically based on the Crank Nicholson scheme, and examined stability and numerical solution convergence.

In the spectral collocation method, Bhrawy and Zaky [25] have developed an algorithm to approximate the numerical solution for 1 and 2-dimensional VOF nonlinear cable equations and described Caputo derivatives. Moghaddam *et al*. [26] submitted a numerical method based upon the VOF delay differential equations scheme for Adams Bashforth-Moulton; the derivatives of VOF were considered in the Caputo sense. They have proven that the solutions exist and are unique.

Li and Wu [27] have constructed a numerical method that uses a kernel theory to solve VOF problems for differential functional equations. Applied for approximate VOF integrals and extended to solve a class of non-linear variable-order fractional equations by delaying, the scheme described by Yaghoobi *et al*. [28] is based on the cubic spline Interpolation.

In this paper we are interested to study a class of time variable-order fractional nonlinear Schrödinger equations (tVOFNLSEs) with the variable-order fractional derivative is defined in the Caputo sense. The construct of the solution rests mainly on the Adomian decomposition method. Numerous numerical examples are presented. Moreover, some solutions are presented graphically when the variable order derivative is given as a specified function.

The structure of this paper is as follows: In section 2, we introduce some basic definitions of the VO-calculus theory. Description of the method is introduced in section 3. The tVOFNLSEs are solved in section 4. Finally, some conclusions and discussions are given.

Preliminaries

We first remember certain definitions and preliminaries of differential and comprehensive VOF operators.

Definition 1: The Riemann-Liouville integral operator of constant order.

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\, f(s)\,ds, \quad \alpha \in R_+ \tag{1}$$

where $\Gamma(.)$ represents the Euler gamma function. For $x > 0$, we have

$$I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta} \tag{2}$$

Definition 2: The Riemann-Liouville and Caputo differential operators of constant order when $n-1 \le \gamma < n,$ of $f(x)$ are given respectively

$$ {}_0D_x^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_0^x (x-s)^{\gamma-n+1} f(s)\, ds, \tag{3} $$

$$ {}_0^C D_x^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x (x-s)^{\gamma-n+1} f^{(n)}(s)\, ds, \tag{4} $$

For $x > 0,$ we have

$$ D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \tag{5} $$

Definition 3: The Riemann-Liouville integral operator of variable fractional order

$$ {}_x I_a^{g(\xi)} f(x) = \frac{1}{\Gamma(g(\xi))} \int_0^x (x-s)^{g(\xi)-1} f(s)\, ds, \quad 0 < g(\xi) \le 1. \tag{6} $$

For $x > 0,$ we have

$$ I_x^{g(\xi)} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+g(\xi)+1)} x^{\beta+g(\xi)} \tag{7} $$

Definition 4: The Riemann-Liouville and Caputo differential operators of variable order when $n-1 \le g_{\min} < g(\xi) < g_{\max} < n,\ n \in \mathrm{N},$ of $f(x)$ are given respectively

$$ {}_0D_x^{g(\xi)} f(x) = \frac{1}{\Gamma(n-g(\xi))} \frac{d^n}{dx^n} \int_0^x (x-s)^{g(\xi)-n+1} f(s)\, ds, \tag{8} $$

$$ {}_0^C D_x^{g(\xi)} f(x) = \frac{1}{\Gamma(n-g(\xi))} \int_0^x (x-s)^{g(\xi)-n+1} f^{(n)}(s)\, ds, \tag{9} $$

For $x > 0,$ we have

$$ D_x^{g(\xi)} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-g(\xi)+1)} x^{\beta-g(\xi)} \tag{10} $$

Adomian decomposition method

We consider the generalized tVOFNLSE of the form

$$ i\frac{\partial^{\alpha(\xi)} u}{\partial t^{\alpha(\xi)}} + a\frac{\partial^2 u}{\partial x^2} + V(x)u + b|u|^2 u = 0, \quad t > 0, \quad 0 < \alpha(\xi) \le 1,\ u(x,0) = f(x). \tag{11} $$

The tVOFNLSE can be written in the operator form as

$$ D_t^{\alpha(\xi)} u = i(a L_x u + V(x)u + b|u|^2 u) \tag{12} $$

where $L_x = \frac{\partial^2}{\partial x^2}$, and $D_t^{\alpha(\xi)} = \frac{\partial^{\alpha(\xi)}}{\partial t^{\alpha(\xi)}}$ is the VOF differential operator. Operating within both sides of Eq. (12) we get

$$u(x,t) = u(x,0) + i\,J^{\alpha(\xi)}(a\,L_x u + V(x)u + b|u|^2\,u), \tag{13}$$

where $J^{\alpha(\xi)} = \int_0^t (\;)dt^{\alpha(\xi)}$ and the nonlinear terms is decomposed as

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n), \tag{14}$$

where A_n is called Adomian's polynomial.
The ADM series solution for $u(x,t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{15}$$

substituting (15) and (14) into (13) yield

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x,0) + iJ^{\alpha(\xi)}\left(\left(a\,L_x \sum_{n=0}^{\infty} u_n\right) + V(x)\left(\sum_{n=0}^{\infty} u_n\right) + b\left(\sum_{n=0}^{\infty} A_n\right)\right), \tag{16}$$

identifying the zeros components $u_0(x,t)$ by $u_0(x,0)$, the remaining components where $n \ge 0$ can be determined by using recurrence relation:

$$u_0(x,t) = u_0(x,0), \qquad u_{n+1}(x,t) = i\,J^{\alpha(\xi)}(a\,L_x u_n + V(x)u_n + b\,A_n), \quad n \ge 0, \tag{17}$$

from this equation, the iterates are defined by the following recursive way

$$\begin{aligned} u_0(x,t) &= u_0(x,0), \\ u_1(x,t) &= i\,J^{\alpha(\xi)}(a\,L_x u_0 + V(x)u_0 + b\,A_0), \\ u_2(x,t) &= i\,J^{\alpha(\xi)}(a\,L_x u_1 + V(x)u_1 + b\,A_1), \\ u_3(x,t) &= i\,J^{\alpha(\xi)}(a\,L_x u_2 + V(x)u_2 + b\,A_2), \end{aligned} \tag{18}$$

where A_n are Adomain's polynomials, which are derived as

$$\begin{aligned} A_0 &= |u_0|^2\,u_0, \\ A_1 &= 2|u_0|^2\,u_1 + u_0^2\overline{u_1}, \\ A_2 &= 2|u_0|^2\,u_2 + u_1^2\overline{u_0} + |u_1|^2\,u_0 + u_0^2\overline{u_2}, \\ A_3 &= 2|u_0|^2\,u_3 + 2u_2u_1\overline{u_0} + 2u_2\overline{u_1}u_0 + |u_1|^2\,u_1 + 2u_0u_1\overline{u_2} + u_0^2\overline{u_3}, \end{aligned} \tag{19}$$

and so on.

Applications

Application 1: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{20}$$

where $\alpha(\xi)$ is an arbitrary function. Eq. (20) models a laser signal propagating in a fiber link under the effect of the dispersion and nonlinearities of the fiber core material. Consider the initial condition

$$u(x,0)=a\,e^{i\frac{c}{2}x}\,\mathrm{sech}(kx), \quad k=\frac{a}{\sqrt{2}},\ a=\sqrt{2(w-0.25)c^2}, \tag{21}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\left(L_x u_n+A_n\right),\ n=0,1,2,\dots$, we can determine some of the terms of the decomposition series (15) as:

$$u_0(x,t)=f(x), u_1(x,t)=\frac{f_1(x)t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ u_2(x,t)=\frac{f_2(x)t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\ u_3(x,t)=\frac{f_3(x)t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)}, \tag{22}$$

where

$$\begin{aligned}&f(x)=a\,e^{i\frac{c}{2}x}\,\mathrm{sech}(kx), \quad f_1(x)=f^{(2)}+|f|^2 f,\\ &f_2(x)=f_1^{(2)}+2|f|^2 f_1+f^2\overline{f_1},\ \ f_3(x)=f_2^{(2)}+2|f|^2 f_2+f_1^2\overline{f}+|f_1|^2 f+f^2\overline{f_2},\end{aligned} \tag{23}$$

and so on. Substituting $u_0,u_1,u_2,u_3,\dots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u(x,t)=f(x)+f_1(x)\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)}+f_2(x)\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}+f_3(x)\frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)}+\dots \tag{24}$$

For the special case $\alpha(\xi)=\alpha$, we have the same solution obtained in [6]

$$u(x,t)=f(x)+f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots \tag{25}$$

another special case when $\alpha=1$, we have the well-known solution

$$u(x,t)=a\,i\,e^{i\left(\frac{c}{2}x+wt-\frac{c^2}{2}t\right)}\mathrm{sech}(k(x-ct)). \tag{26}$$

Application 2: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+\frac{\partial^2 u}{\partial x^2}+2|u|^2 u=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{27}$$

This example resembles example 1 but the only difference is doubling the nonlinearity term. Considering the initial condition;

$$u(x,0)=e^{ix}, \tag{28}$$

using $u_0(x,t) = u_0(x,0)$ and $u_{n+1}(x,t) = i\, J^{\alpha(\xi)}\left(L_x u_n + 2A_n\right),\ n = 0,1,2,\dots,$ we can determine some of the terms of the decomposition series (15) as:

$$u_0 = e^{ix},\ \ u_1 = \frac{B_1 e^{ix} t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ \ u_2 = \frac{B_2 e^{ix} t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\ \ u_3 = \frac{B_3 e^{ix} t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ u_4 = \frac{B_4 e^{ix} t^{4\alpha(\xi)}}{\Gamma(4\alpha(\xi)+1)},\dots \tag{29}$$

and so on. substituting $u_0, u_1, u_2, u_3, \dots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u = e^{ix}\left(1 + \frac{B_1\, t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)} + \frac{B_2\, t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)} + \frac{B_3\, t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)} + \dots\right) = e^{ix}\left(1 + \sum_{k=1}^{\infty} \frac{B_k\, t^{k\alpha(\xi)}}{\Gamma(k\alpha(\xi)+1)}\right), \tag{30}$$

where

$$B_1 = i\left(e^{i\pi} + 2\right), \qquad B_2 = i\left(e^{i\pi} B_1 + 2\left(\overline{B_1} + 2B_1\right)\right),$$

$$B_3 = i\left(e^{i\pi} B_2 + 2\left[\overline{B_2} + 2B_2 + \frac{\Gamma(2\alpha(\xi)+1)}{(\Gamma(\alpha(\xi)+1))^2}\left(2\,|\,B_1\,|^2 + B_1^2\right)\right]\right),\dots$$

and so on. We are going to discuss the solution given by Eq. (30) to understand the effect of the fractional variable order $\alpha(\xi)$. The evolution behavior of real and imaginary parts of the solution (30) in different choices of $\alpha(\xi)$. **Figures 1**(**a**) - **1**(**h**) represented the real and imaginary parts of (30) at fixed $x = 5$ for $\alpha(\xi) = 1 - \frac{\sin^2 \xi}{3},\ \ \frac{10 - 2\sin\xi}{12}, \frac{2 - e^{-\xi}}{2}, \frac{\xi}{\xi+1}$ respectively. From **Figure 1**, we see that the evolution behavior of the solution (30) depends on the choices of the variable order $\alpha(\xi)$. When $\alpha(\xi) = \alpha$, we have the same solution obtained in [6]

$$u = e^{ix}\left(1 + \frac{C_1\, t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_2\, t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{C_3\, t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right) = e^{ix}\left(1 + \sum_{k=1}^{\infty} \frac{C_k\, t^{k\alpha}}{\Gamma(k\alpha+1)}\right), \tag{31}$$

where

$$C_1 = i\left(e^{i\pi} + 2\right), \qquad C_2 = i\left(e^{i\pi} B_1 + 2\left(\overline{B_1} + 2B_1\right)\right),$$

$$C_3 = i\left(e^{i\pi} B_2 + 2\left[\overline{B_2} + 2B_2 + \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\left(2\,|\,B_1\,|^2 + B_1^2\right)\right]\right),\dots$$

As $\alpha = 1$, we have $B_1 = i,\ B_2 = i^2,\ B_3 = i^3,\dots$ and the solution (30) takes the following form

$$u = e^{ix}\left(1 + \frac{i\,t}{\Gamma(2)} + \frac{i^2\, t^2}{\Gamma(3)} + \frac{i^3\, t^3}{\Gamma(4)} + \dots\right) = e^{ix}\left(\sum_{k=1}^{\infty} \frac{(i\,t)^k}{k!}\right).$$

Using Taylor series expansion near $t = 0$, we have:

$$u(x,t) = e^{i(x+t)}. \tag{32}$$

which is the well-known solution of the NLS equation.

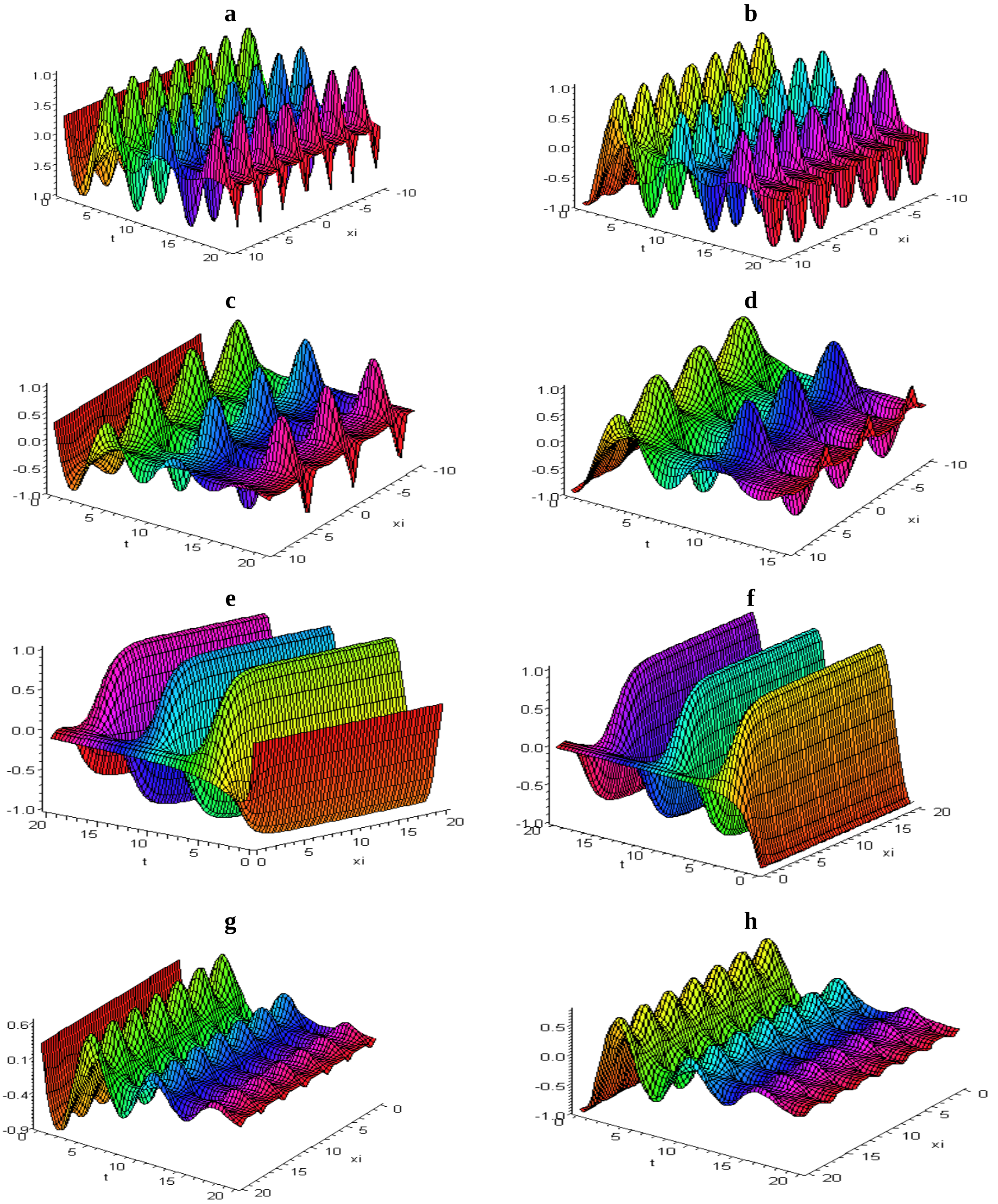

Figure 1 Evolution behavior of real and imaginary parts of Eq. (30): (a) and (b) when $\alpha(\xi)=1-\sin^2(\xi)/3$. (c) and (d) when $\alpha(\xi)=(10-2\sin\xi)/12$. (e) and (f) when $\alpha(\xi)=1-e^{-\xi}/2$. (g) and (h) when $\alpha(\xi)=(14+\sin(2\xi)/16$.

Application 3: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u\cos^2 x-|u|^2 u=0,\quad t>0,\quad 0<\alpha(\xi)\le 1, \tag{33}$$

Eq. (33) models a solitary wave under the effect of dispersion, nonlinearity, and periodic trapping potential ($\cos^2 x$). Providing the initial condition

$$u(x,0)=\sin x, \tag{34}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\left(\frac{1}{2}L_x u_n-u_n\cos^2 x-A_n\right),\quad n=0,1,2,\ldots$, we can calculate some of the terms of the decomposition series (15) as:

$$u_0=\sin x,\ \ u_1=\left(-\frac{3i}{2}\right)\frac{t^{\alpha(\xi)}\sin x}{\Gamma(\alpha(\xi)+1)},\ \ u_2=\left(-\frac{3i}{2}\right)^2\frac{t^{2\alpha(\xi)}\sin x}{\Gamma(2\alpha(\xi)+1)},$$
$$u_3=\left(-\frac{3i}{2}\right)^3\frac{t^{3\alpha(\xi)}\sin x}{\Gamma(3\alpha(\xi)+1)},\ \ldots \tag{35}$$

and so on. substituting $u_0,u_1,u_2,u_3,\ldots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u(x,t)=\sin x\left(\sum_{k=0}^{\infty}\left(-\frac{3i}{2}\right)^k\frac{t^{k\alpha(\xi)}\sin x}{\Gamma(k\alpha(\xi)+1)}\right)=\sin x\,E_{\alpha(\xi)}\left(-\frac{3i}{2}t^{\alpha(\xi)}\right), \tag{36}$$

where $E_{\alpha(\xi)}(t^{\alpha(\xi)})$ is the Mittag-Leffler function in 1 variable parameter $\alpha(\xi)$. The generalized trigonometric functions with variable parameters $\alpha(\xi)$ are defined as

$$\sin(t,\alpha(\xi))=\frac{E_{\alpha(\xi)}(i\,t^{\alpha(\xi)})-E_{\alpha(\xi)}(-i\,t^{\alpha(\xi)})}{2i},\ \ \cos(t,\alpha(\xi))=\frac{E_{\alpha(\xi)}(i\,t^{\alpha(\xi)})+E_{\alpha(\xi)}(-i\,t^{\alpha(\xi)})}{2},$$

so Eq. (35) can be written in the following form

$$u(x,t)==\sin x\left(\cos\left(1.5t,\alpha(\xi)\right)-i\sin\left(1.5t,\alpha(\xi)\right)\right).$$

Evolution behavior of the real and imaginary parts of Eq. (36) at fixed $x=10$ are represented in **Figure 2** when the variable order $\alpha(\xi)=\frac{9-\cos\xi}{12},\frac{14-2\sin\xi}{16},\ \frac{5-\sin\xi}{6},\ \frac{2+\cos\xi}{3}$. When $\alpha(\xi)=\alpha$, we have the same solution obtained in [6]

$$u(x,t)=\sin x\,E_\alpha\left(-\frac{3i\,t^\alpha}{2}\right), \tag{37}$$

As $\alpha=1$, we have

$$u(x,t)=\sin x\ e^{-\frac{3it}{2}}, \tag{38}$$

which is the well-known exact solution.

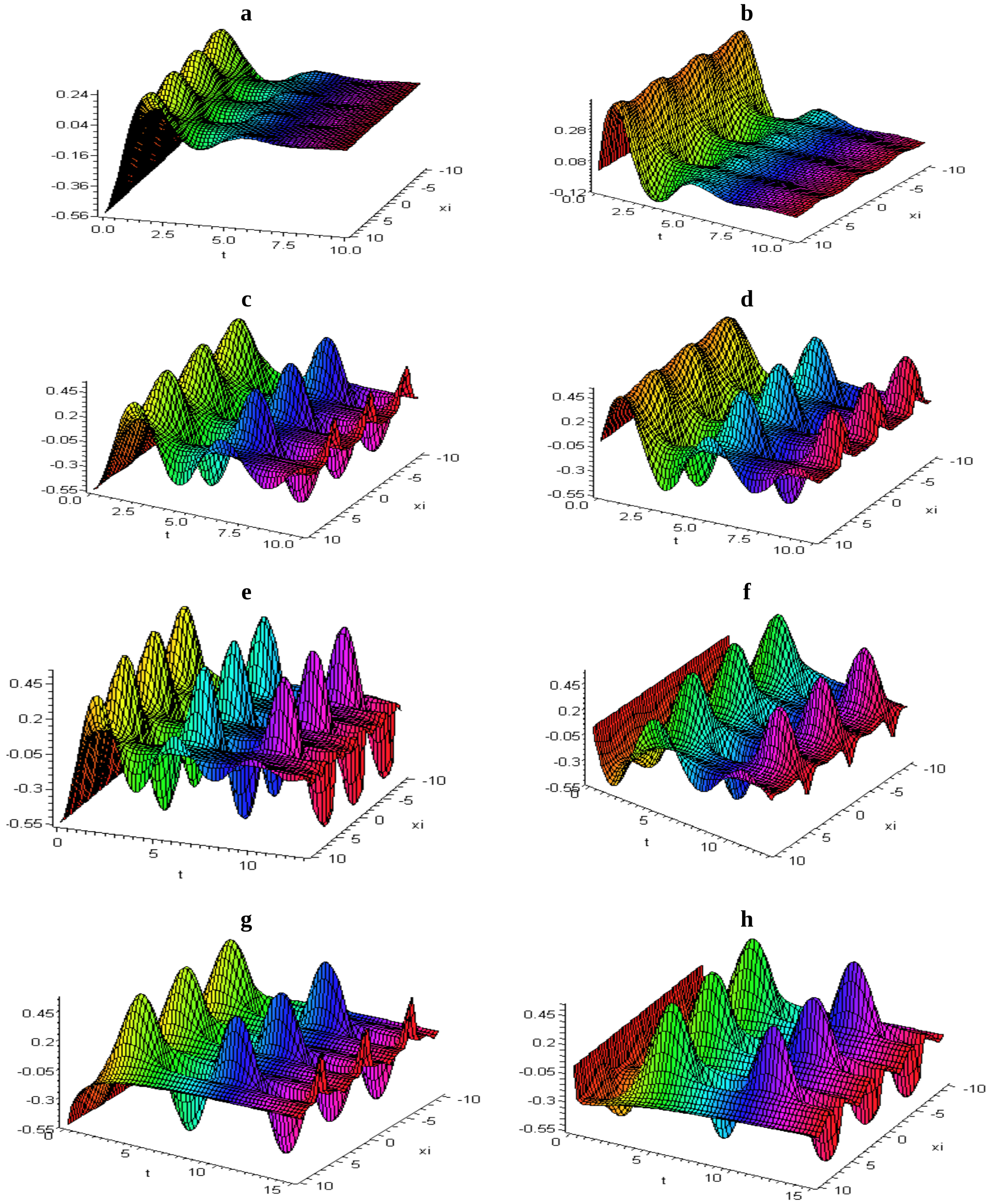

Figure 2 Evolution behavior of real and imaginary parts of Eq. (36): (a) and (b) when $\alpha(\xi)=(9-\cos\xi)/12$. (c) and (d) when $\alpha(\xi)=(14-2\sin\xi)/16$. (e) and (f) when $\alpha(\xi)=(5-\sin\xi)/6$. (g) and (h) when $\alpha(\xi)=(2+\cos\xi)/3$.

Application 4: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u\cos^2 x+|u|^2 u=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{39}$$

application 4 is the same as application 3 except for reverse sign nonlinearity. Providing the initial condition

$$u(x,0)=\cos x, \tag{40}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\left(\frac{1}{2}L_x u_n-u_n\cos^2 x+A_n\right),\ n=0,1,2,\ldots$, we can calculate some of the terms of the decomposition series (15) as:

$$\begin{aligned}&u_0=\cos x, \qquad u_1=-\frac{i}{2}\cos x\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ u_2=\left(-\frac{i}{2}\right)^2\cos x\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}\\&u_3=\left(-\frac{i}{2}\right)^3\cos x\ \frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ldots\end{aligned} \tag{41}$$

and so on. substituting $u_0,u_1,u_2,u_3,\ldots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u(x,t)=\cos x\,E_{\alpha(\xi)}\left(-\frac{i}{2}t^{\alpha(\xi)}\right)=\cos x[\cos(0.5t,\alpha(\xi))-i\sin(0.5t,\alpha(\xi))], \tag{42}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=\cos x\,E_{\alpha}\left(-\frac{i\,t^{\alpha}}{2}\right), \tag{43}$$

As $\alpha=1$, we have

$$u(x,t)=\cos x\ e^{-\frac{it}{2}}, \tag{44}$$

which is the well-known exact solution.

Application 5: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+\frac{\partial^2 u}{\partial x^2}-2|u|^2 u=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{45}$$

This looks like example 2 but with positive dispersion. Considering the initial condition

$$u(x,0)=e^{ix}, \tag{46}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\left(L_x u_n-2A_n\right),\ n=0,1,2,\ldots$, we can determine some of the terms of the decomposition series (15) as:

$$u_0 = e^{ix},\;\; u_1 = \frac{B_1 e^{ix} t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\;\; u_2 = \frac{B_2 e^{ix} t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\;\; u_3 = \frac{B_3 e^{ix} t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\;\; u_4 = \frac{B_4 e^{ix} t^{4\alpha(\xi)}}{\Gamma(4\alpha(\xi)+1)},\ldots \quad (47)$$

and so on. substituting $u_0, u_1, u_2, u_3, \ldots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u = e^{ix}\left(1 + \frac{B_1 t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)} + \frac{B_2 t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)} + \frac{B_3 t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)} + \ldots\right) = e^{ix}\left(1 + \sum_{k=1}^{\infty} \frac{B_k t^{k\alpha(\xi)}}{\Gamma(k\alpha(\xi)+1)}\right), \quad (48)$$

where

$$B_1 = i\left(e^{i\pi} - 2\right), \qquad B_2 = i\left(e^{i\pi} B_1 - 2\left(\overline{B_1} + 2B_1\right)\right),$$
$$B_3 = i\left(e^{i\pi} B_2 - 2\left[\overline{B_2} + 2B_2 + \frac{\Gamma(2\alpha(\xi)+1)}{(\Gamma(\alpha(\xi)+1))^2}\left(2|B_1|^2 + B_1^2\right)\right]\right),\ldots$$

and so on. When $\alpha(\xi) = \alpha$, we have

$$u = e^{ix}\left(1 + \frac{C_1 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{C_3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) = e^{ix}\left(1 + \sum_{k=1}^{\infty} \frac{C_k t^{k\alpha}}{\Gamma(k\alpha+1)}\right), \quad (49)$$

where

$$C_1 = i\left(e^{i\pi} - 2\right), \qquad C_2 = i\left(e^{i\pi} B_1 - 2\left(\overline{B_1} + 2B_1\right)\right),$$
$$C_3 = i\left(e^{i\pi} B_2 - 2\left[\overline{B_2} + 2B_2 + \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\left(2|B_1|^2 + B_1^2\right)\right]\right),\ldots$$

As $\alpha = 1$, we have $B_1 = -3i,\ B_2 = (-3i)^2,\ B_3 = (-3i)^3,\ldots$ and the solution (46) takes the following form

$$u = e^{ix}\left(1 + \frac{(-3i)t}{\Gamma(2)} + \frac{(-3i)^2 t^2}{\Gamma(3)} + \frac{(-3i)^3 t^3}{\Gamma(4)} + \ldots\right) = e^{ix}\left(\sum_{k=1}^{\infty} \frac{(-3i\,t)^k}{k!}\right).$$

Using Taylor series expansion near $t = 0$, we have:

$$u(x,t) = e^{i(x-3t)}. \quad (50)$$

Example 6: Consider the following tVOFNLSE

$$i\frac{\partial^{\alpha(\xi)} u}{\partial t^{\alpha(\xi)}} - \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \qquad t > 0, \qquad 0 < \alpha(\xi) \le 1, \quad (51)$$

this resembles example 5 but with negative dispersion and positive nonlinearity. Consider the initial condition

$$u(x,0) = e^{ix}, \quad (52)$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=-i\,J^{\alpha(\xi)}\left(L_x u_n-2A_n\right),\ n=0,1,2,\ldots$, we can determine some of the terms of the decomposition series (15) as:

$$u_0=e^{ix},\ \ u_1=\frac{B_1e^{ix}t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ \ u_2=\frac{B_2e^{ix}t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\ \ u_3=\frac{B_3e^{ix}t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ u_4=\frac{B_4e^{ix}t^{4\alpha(\xi)}}{\Gamma(4\alpha(\xi)+1)},\ldots \quad (53)$$

and so on. substituting $u_0,u_1,u_2,u_3,\ldots$ into Eq. (15) gives the solution $u(x,t)$ in a series form by

$$u=e^{ix}\left(1+\frac{B_1\,t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)}+\frac{B_2\,t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}+\frac{B_3\,t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)}+\ldots\right)=e^{ix}\left(1+\sum_{k=1}^{\infty}\frac{B_k\,t^{k\alpha(\xi)}}{\Gamma(k\alpha(\xi)+1)}\right), \quad (54)$$

where

$$B_1=-i\left(e^{i\pi}-2\right),\qquad B_2=-i\left(e^{i\pi}B_1-2\left(\overline{B_1}+2B_1\right)\right),$$
$$B_3=-i\left(e^{i\pi}B_2-2\left[\overline{B_2}+2B_2+\frac{\Gamma(2\alpha(\xi)+1)}{(\Gamma(\alpha(\xi)+1))^2}\left(2\,|\,B_1\,|^2+B_1^2\right)\right]\right),\ldots$$

and so on. When $\alpha(\xi)=\alpha$, we have

$$u=e^{ix}\left(1+\frac{C_1\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{C_2\,t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{C_3\,t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right)=e^{ix}\left(1+\sum_{k=1}^{\infty}\frac{C_k\,t^{k\alpha}}{\Gamma(k\alpha+1)}\right), \quad (55)$$

where

$$C_1=-i\left(e^{i\pi}-2\right),\qquad C_2=-i\left(e^{i\pi}B_1-2\left(\overline{B_1}+2B_1\right)\right),$$
$$C_3=-i\left(e^{i\pi}B_2-2\left[\overline{B_2}+2B_2+\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\left(2\,|\,B_1\,|^2+B_1^2\right)\right]\right),\ldots$$

As $\alpha=1$, we have $B_1=3i,\ B_2=(3i)^2,\ B_3=(3i)^3,\ldots$ and the solution (54) takes the following form

$$u=e^{ix}\left(1+\frac{3i\,t}{\Gamma(2)}+\frac{(3i)^2\,t^2}{\Gamma(3)}+\frac{(3i)^3\,t^3}{\Gamma(4)}+\ldots\right)=e^{ix}\left(\sum_{k=1}^{\infty}\frac{(3i\,t)^k}{k!}\right).$$

Using Taylor series expansion near $t=0$, we have:

$$u(x,t)=e^{i(x+3t)}. \quad (56)$$

Application 7: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+i\frac{\partial^2 u}{\partial x^2}=0,\qquad t>0,\qquad 0<\alpha(\xi)\le 1, \quad (57)$$

Eq. (57) models a solitary wave under the effect of dispersion only. Providing it is subject to the initial condition

$$u(x,0)=e^{3ix}, \tag{58}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=-i\,J^{\alpha(\xi)}\,L_{xx}u_n,\quad n=0,1,2,...$, we drive:

$$u_0=e^{3ix},\ \ u_1=\frac{(9i)\ t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)}\ e^{3ix},\ \ u_2=\frac{(9i)^2\ t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}e^{3ix},\ u_3=\frac{(9i)^3\ t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)}\ e^{3ix},\ \ ... \tag{59}$$

Hence

$$u=e^{3ix}E_{\alpha(\xi)}\left(9i\,t^{\alpha(\xi)}\right), \tag{60}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=e^{3ix}E_{\alpha}\left(9i\,t^{\alpha}\right), \tag{61}$$

As $\alpha=1$, we have

$$u(x,t)=e^{3i(x+3t)}. \tag{62}$$

Application 8: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+i\frac{\partial^2 u}{\partial x^2}=0,\quad t>0,\quad 0<\alpha(\xi)\le 1, \tag{63}$$

which is the same as example 7 but the initial condition is

$$u(x,0)=\cosh(x), \tag{64}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=-i\,J^{\alpha(\xi)}\,L_{xx}u_n,\quad n=0,1,2,...$, we drive:

$$\begin{aligned}&u_0=\cosh(x),\ u_1=(-i)\cosh(x)\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ u_2=(-i)^2\cosh(x)\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\\&u_3=(-i)^3\cosh(x)\frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ ...\end{aligned} \tag{65}$$

Hence

$$u(x,t)=\cosh(x)E_{\alpha(\xi)}\left(-i\,t^{\alpha(\xi)}\right)=\cosh(x)\cos\left(t,\alpha(\xi)\right)-i\cosh(x)\sin\left(t,\alpha(\xi)\right). \tag{66}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=\cosh(x)E_{\alpha}\left(-i\,t^{\alpha}\right) \tag{67}$$

As $\alpha=1$, we have

$$u(x,t)=\cosh(x)e^{-it}. \tag{68}$$

Application 9: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}+i\frac{\partial^2 u}{\partial x^2}=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{69}$$

which is the same as application 7 but the initial condition is

$$u(x,0)=\cos(x), \tag{70}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=-i\,J^{\alpha(\xi)}\,L_{xx}u_n, \quad n=0,1,2,\ldots$, we drive:

$$\begin{gathered} u_0=\cos(x),\; u_1=i\cos(x)\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\; u_2=(i)^2\cos(x)\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}, \\ u_3=(i)^3\cos(x)\frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\; \ldots \end{gathered} \tag{71}$$

Hence

$$u(x,t)=\cos(x)E_{\alpha(\xi)}\left(i\,t^{\alpha(\xi)}\right)=\cos(x)\cos\left(t,\alpha(\xi)\right)+i\cos(x)\sin\left(t,\alpha(\xi)\right), \tag{72}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=\cos(x)E_{\alpha}\left(i\,t^{\alpha}\right) \tag{73}$$

As $\alpha=1$, we have

$$u(x,t)=\cos(x)e^{it}. \tag{74}$$

Application 10: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}-i\frac{\partial^2 u}{\partial x^2}=0, \quad t>0, \quad 0<\alpha(\xi)\le 1, \tag{75}$$

This resembles a**pplication 7** but with negative dispersion only. Assuming the initial condition

$$u(x,0)=\sinh(x), \tag{76}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\,L_{xx}u_n, \quad n=0,1,2,\ldots$, we drive:

$$\begin{gathered} u_0=\sinh(x),\; u_1=(i)\sinh(x)\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\; u_2=(i)^2\sinh(x)\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)}, \\ u_3=(i)^3\sinh(x)\frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\; \ldots \end{gathered} \tag{77}$$

Hence

$$u(x,t)=\sinh(x)E_{\alpha(\xi)}\left(i\,t^{\alpha(\xi)}\right)=\cosh(x)\cos\left(t,\alpha(\xi)\right)+i\cosh(x)\sin\left(t,\alpha(\xi)\right), \tag{78}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=\sinh(x)E_{\alpha}\left(i\,t^{\alpha}\right) \tag{79}$$

As $\alpha=1$, we have

$$u(x,t)=\sinh(x)e^{it}. \tag{80}$$

Application 11: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}-i\frac{\partial^{2}u}{\partial x^{2}}=0,\quad t>0,\quad 0<\alpha(\xi)\le 1, \tag{81}$$

which is the same as **application 10** but subject to the initial condition

$$u(x,0)=\sin(x), \tag{82}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\,L_{xx}u_n,\quad n=0,1,2,\ldots$, we drive:

$$\begin{aligned}&u_0=\sin(x),\ u_1=-i\sin(x)\frac{t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ u_2=(-i)^2\sin(x)\frac{t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\\&u_3=(-i)^3\sin(x)\frac{t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ldots\end{aligned} \tag{83}$$

Hence

$$u(x,t)=\sin(x)E_{\alpha(\xi)}\left(-i\,t^{\alpha(\xi)}\right)=\sin(x)\cos\left(t,\alpha(\xi)\right)-i\sin(x)\sin\left(t,\alpha(\xi)\right), \tag{84}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t)=\sin(x)E_{\alpha}\left(-i\,t^{\alpha}\right) \tag{85}$$

As $\alpha=1$, we have

$$u(x,t)=\sin(x)e^{-it}. \tag{86}$$

Application 12: Consider the following linear tVOFSE

$$\frac{\partial^{\alpha(\xi)}u}{\partial t^{\alpha(\xi)}}-i\frac{\partial^{2}u}{\partial x^{2}}=0,\quad t>0,\quad 0<\alpha(\xi)\le 1, \tag{87}$$

which is the same as **application 10** but subject to the initial condition

$$u(x,0)=e^{ix}, \tag{88}$$

using $u_0(x,t)=u_0(x,0)$ and $u_{n+1}(x,t)=i\,J^{\alpha(\xi)}\,L_{xx}u_n,\quad n=0,1,2,\ldots$, we drive:

$$u_0 = e^{ix},\ u_1 = -\frac{ie^{ix}\,t^{\alpha(\xi)}}{\Gamma(\alpha(\xi)+1)},\ u_2 = \frac{(-i)^2\,e^{ix}t^{2\alpha(\xi)}}{\Gamma(2\alpha(\xi)+1)},\ u_3 = \frac{(-i)^3\,e^{ix}t^{3\alpha(\xi)}}{\Gamma(3\alpha(\xi)+1)},\ \ldots \tag{89}$$

Hence

$$u(x,t) = e^{ix}E_{\alpha(\xi)}\left(-i\,t^{\alpha(\xi)}\right), \tag{90}$$

When $\alpha(\xi)=\alpha$, we have

$$u(x,t) = e^{ix}E_{\alpha}\left(-i\,t^{\alpha}\right) \tag{91}$$

As $\alpha=1$, we have

$$u(x,t) = e^{i(x-t)}. \tag{92}$$

Conclusions and discussions

The NLSE plays an important model in different fields; for example, it describes solitary wave propagation in a fiber link and water, waves in Bose-Einstein condensation, and plasma physics. Some examples are given to illustrate the solutions of linear and nonlinear variable order fractional Schrödinger equations using the Adomian decomposition method. As far as we know, the fractional-order (constant or standard integer derivative order) is solved for the 1st time in the literature. We have introduced a new procedure for solving the tVOFNLSE and tVOFSE. This algorithm is developed and extended to obtain analytical solutions of the class of tVOFNLSE and tVOFSE.

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