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Application of Iterative Three-Dimensional Laplace Transform Method for 2-Dimensional Nonlinear Klein-Gordon Equation

Alemayehu Tamirie Deresse

Department of Mathematics, Faculty of Natural Sciences, Mizan Tepi University, Tepi 260, Ethiopia

(Corresponding author's e-mail: alemayehutamirie@mtu.edu.et)

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Abstract

In the present study, the exact analytical solutions of the 2-dimensional nonlinear Klein-Gordon equation (NLKGE) is investigated using the 3-dimensional Laplace transform method in conjunction with the Daftardar-Gejji and Jafari Method (Iterative method). Through this method, the linear part of the problem is solved by using the 3-dimensional Laplace transform method, while the noise terms from the nonlinear part of the equation disappear through a successive iteration process of the Daftardar-Gejji and Jafari Method (DJM), where a single iteration gives the exact solution to the problem. Three test modeling problems from mathematical physics nonlinear Klein-Gordon equations are taken to confirm the performance and efficiency of the presented technique. For each illustrative example, the convergence of the novel iterative approach is shown. The findings also suggest that the proposed method could be applied to other types of nonlinear partial differential equation systems.

Keywords: Nonlinear Klein-Gordon equation, 3-dimensional Laplace transform, Inverse Laplace transform, DJM method, Convergence of DJM method

Introduction

Partial differential equations are tremendously essential in mathematics and other fields of study. Nonlinear partial differential equations (PDEs) play a crucial role in the formulation of fundamental laws of nature and the mathematical analysis of a wide range of issues in applied mathematics, mathematical physics, and engineering science. In a range of scientific domains, such as solid-state physics, fluid dynamics, plasma physics, mathematical biology, and chemical kinematics, PDEs can be utilized to mimic nonlinear processes [1,2]. The Klein-Gordon equation (KGE) is a nonlinear hyperbolic class of partial differential equations that occur in relativistic quantum mechanics and field theory, both of which are crucial to high-energy physicists [3].

The Klein-Gordon equation (KGE) is a relativistic variation of the Schrödinger equation for free particles, devised and named after scientists Oskar Klein and Walter Gordon in 1926. This equation is used to model a variety of physical problems, such as the nonlinear optics, the nuclear physics, the magnetic field theory, the propagation of fluxons between 2 superconductors in Josephson junctions, solid-state physics, mathematical biology, chemical kinematics, the motion of rigid pendula attached to a stretched wire, the behavior of elementary particles, and dislocation propagation in crystals [4,5].

Since KGE is one of the crucial equations in nonlinear science, different researchers developed Several new analytical/numerical methods to accomplish the solutions of these NKGEs, for instance, Changa and Kuo [7] introduced a new method which is called a Lie-group approach for solving the backward 2-dimensional nonlinear Klein-Gordon equations. To solve the D-dimensional Klein-Gordon equation with equal scalar and vector potentials, Okorie *et al.* [8] proposed a novel generalized hyperbolic potential model. The relativistic bound state energy equation was discovered using the functional analysis method. They also identified the ro-vibrational energy spectrum of numerous diatomic substances in both relativistic and non-relativistic states. Siddique *et al.* [9] used the (G'/G)-expansion method to investigate the traveling wave solutions of the (2+1)-Klein-Gordon equation involving quadratic, cubic, and polynomial nonlinearity. The targeted traveling wave solutions are found in the form of hyperbolic, trigonometric, and rational functions using the presented technique. The authors of the paper [49] solve the D-dimensional Klein-Gordon equation with a generalized hyperbolic potential model, under the condition of equal scalar and vector potentials. They obtained the relativistic bound state energy equation by using the functional analysis method. Also, they obtained the relativistic and non-relativistic ro-vibrational energy spectra for different diatomic molecules. To get the approximate solutions to the nonlinear Klein-Gordon equations, the authors of the research [50] used the perturbation iteration approach and a newly designed optimal perturbation iteration method. They also explained how some partial differential equations' optimal perturbation iteration method converges.

In 2022, Kumar [51] proposed a numerical technique based upon Haar scale 3 wavelets for the solution of non-linear Klein-Gordon equations. In this technique, spatial discretization is done by Haar scale 3 Wavelets. The collocation method is used with Haar scale 3 wavelets to convert Klein-Gordon equation into the system of algebraic equations. The author of the paper [10] used a meshless approach to solve a class of nonlinear generalized Klein-Gordon equations numerically. The temporal derivatives are discretized using a time-discrete approach, and the spatial derivatives are approximated using a well-posed moving least squares (WP-MLS) approximation using shifted and scaled orthogonal basis functions. Moreover, the author provided an iterative scheme within its corresponding convergence to overcome the complexity of the nonlinearity in solving the problem. Another study conducted by Belayeh *et al.* [11], used the reduced differential transform method (RDTM) to solve (2+1)-dimensional nonlinear Klein-Gordon equations (NLKGEs) with quadratic and cubic nonlinearities as long as the initial conditions were met. By this method, the performance and efficiency of the proposed technique are demonstrated by taking 2 test examples from mathematical physics. For more related research about the solution of the KGEs, interested readers may refer to [12-15].

The Laplace Transform Method (LTM) is an integral transform method for solving linear differential, difference, and integral equations that have been widely employed [16,47]. In engineering and physics, the Laplace transform method is widely used to calculate the output of a linear time-invariant system by convolving its unit impulse response with the input signal [48]. When this calculation is done in Laplace space, the convolution becomes a multiplication, which is easier to solve due to its algebraic nature. The Laplace transform converts a linear differential equation into an algebraic equation that can be solved using algebraic techniques. The inverse Laplace transform is then used to solve the original differential equation [17].

There are numerous uses for the Laplace transform throughout actual life sciences. For instance, the Laplace transform is used in physics to solve Newton's Law of Cooling, which predicts the cooling of a heated body placed in a cold environment, and to determine the harmonic vibration of a beam supported at both ends [48]. Displaying of high voltage D.C. (Direct Current) converters dependent on H-connect voltage converters is exhibited by the author of the paper [52]. This has more highlights contrasted with 2-level Converters. It revises power framework phenomena like force quality, voltage, and 1st swing steadiness Plans of proportional Integral (P.I) type compensators are inferred by Laplace Transformations. Hazarika *et al.* [53] have focused on the effect of dot clamor on tasks involving programmed data extraction and SAR images, and this effect is mitigated by using the Laplace Transform Technique. Another effective method is developed for SAR image Despeckling. To despeckle SAR images, a different type of Laplace symmetric transform (LOT) is suggested. See [54] the reference therein for more details.

Researchers extended the concept of the single Laplace transform method to the double Laplace transform method to find the solution to some kinds of differential equations and fractional differential equations, such as space-time fractional telegraph equations and functional, integral, and partial differential equations, because the Laplace transform method is a crucial technique in solving mathematical problems arising in various fields of science [18-20].

Due to its high importance in solving real-life associated differential equations, several researchers have recently extended the double Laplace transform approach to the triple (3-dimensional) Laplace Transform method (TLTM). Khan *et al.* [21] employed the Caputo fractional derivative to discover the solution to fractional order partial differential equations using the 3-dimensional Laplace transform approach. The authors of the paper [22-25] were successfully applied the 3-dimensional Laplace transform method to obtain the solutions of fractional order telegraph equations in 2-dimensions, linear Volterra Integro-differential equations in 3-dimensions, and 3rd-order Mboctara equations, and the proof of some properties of the proposed method like linearity property, change of scale property, 1st shifting property, convolution theorem property, differential property, and 3-dimensional integral property are also presented.

When compared to other known methods, the 3-dimensional Laplace transform method provides rapid convergence of the exact solution without any restrictive assumptions about the solution [26]. However, this transform fails to solve some complex nonlinear partial differential equations that arise in mathematical physics, engineering fields, and other related fields of sciences that involve nonlinear phenomena. In such situations, to solve a similar problem, the Laplace Transform method is often combined with other efficient methods like, He's polynomials [27], Variational Iteration Method [28], Daftardar-Gejji and Jafari Method [29,30], Homotopy Perturbation method [31,32], Adomian decomposition method [33-35], Differential transform method [46], and so on.

Daftardar-Gejji and Jafari [36] introduced the novel iterative technique (NIM) or Daftardar-Gejji and Jafari method (DJM) for solving linear and nonlinear functional equations. The solution of integer and fractional order nonlinear differential equations with linear and nonlinear ordinary and partial differential equations using the DJM is an interesting challenge in applied scientific disciplines [37,38]. The proposed method's iterative technique yields a series that can be summed up to get an analytical formula or used to build an appropriate approximation with a faster convergent series solution [39,40]. The error of the approximation can be controlled by properly truncating the series [41].

To obtain an accurate solution of the 1-dimensional non-linear Klein-Gordon and Telegraph equations, the authors combined the Daftardar-Gejji and Jafari approach with the double Laplace transform method in the work [29,30]. By this method, the noise terms cancel each other in the iteration process, and a single iteration gives the exact solution. Similar research conducted by the authors of the paper [44] combines the double Laplace transform method with the Adomain decomposition method to obtain the approximate analytical solutions of the 1-dimensional homogeneous Klein-Gordon equation. In paper [46] the authors utilized the Laplace transform method in conjunction with the differential transform method (DTM) to solve some nonlinear nonhomogeneous partial differential equations. Deresse *et al.* [42,45] used the 3-dimensional Laplace transform method combined with the Daftardar-Gejji and Jafari Method to find the exact solution of the (2+1)-dimensional nonlinear sine-Gordon equation and Telegraph equation. Through this approach, the non-linear term that emerges in the equation and is difficult to solve using the Laplace transform method is eliminated through a series of iterations of the Daftardar-Gejji and Jafari Method procedures. Therefore, the presented technique gives the exact solution by being easily applied analytically to the given problems, and this is the greatest importance of the presented method. However, this method is not yet studied for the 2-dimensional non-linear Klein-Gordon equation in the existing literature. As a result, the current investigation is prepared to fill this gap.

The goal of this study is to combine 2 powerful techniques, the 3-dimensional Laplace transform method (TLTM) and the Daftardar-Gejji and Jafari method (DJM) or the New iterative method (NIM), in order to find an accurate solution to the 2-dimensional Klein-Gordon equation (NLKGE) with quadratic, cubic, and polynomial nonlinearity under the provided initial and boundary conditions (Cauchy type BCs). Moreover, for each considered illustrative example the corresponding 3D plots of exact solutions for different values of time variable "t" are depicted in **Figures 1 - 3**, respectively.

The n −dimensional nonlinear Klein-Gordon equation is given by a partial differential Eq. [10]:

$$u_{tt}(\boldsymbol{x},t)+\beta u_t(\boldsymbol{x},t)=\alpha\left(u_{xx}(\boldsymbol{x},t)+u_{yy}(\boldsymbol{x},t)\right)-\mathrm{N}\left(u(\boldsymbol{x},t)\right)+f(\boldsymbol{x},t),\quad x\in\Omega,\ t>0, \tag{1}$$

subject to the initial conditions

$$u(\boldsymbol{x},0)=\varphi_1(\boldsymbol{x}),u_t(\boldsymbol{x},0)=\varphi_2(\boldsymbol{x}), \tag{2}$$

and boundary conditions (Cauchy type BCs)

$$u(\boldsymbol{x},t)=\bar{u}(\boldsymbol{x},t),\ \boldsymbol{x}\in\Gamma_D, t>0, \tag{3}$$

$$q(\boldsymbol{x},t)=u_{\boldsymbol{n}}(\boldsymbol{x},t)=\bar{q}(\boldsymbol{x},t),\ \boldsymbol{x}\in\Gamma_N, t>0, \tag{4}$$

provided that Ω is a domain of $\mathbb{R}^n\left(n=1,2,3,...\right)$ with boundary $\Gamma=\Gamma_N+\Gamma_D$, $\boldsymbol{x}=\left(x_1,x_2,x_3,...,x_n\right)^T$ is the space variable t is the time variable, $u(\boldsymbol{x},t)$ is the unknown function, $N(u)$ is a nonlinear function of $u(\boldsymbol{x},t)$, $f(\boldsymbol{x},t)$ is a given function, $\varphi_1(\boldsymbol{x})$ and $\varphi_2(\boldsymbol{x})$ are prescribed initial functions, $\bar{u}(\boldsymbol{x},t)$ and $\bar{q}(\boldsymbol{x},t)$ are prescribed boundary functions, $\boldsymbol{n}=\left(n_1,n_2,n_3,...,n_n\right)^T$ is the unit outward normal on Γ, and α and β are known constants.

In physical applications, the nonlinear force $N(u)$ takes different forms as shown in the **Table 1** below (see [6,43] and references therein);

Table 1 Different forms of the nonlinear force $N(u)$.

N)u(	**Name of Eq. (1)**
$\sin u$	Sine-Gordon equation
$\sinh u$	Sinh-Gordon equation
$\sin u + \sin 2u$	Double sine-Gordon equation
$\sinh u + \sinh 2u$	Double sinh-Gordon equation
$\exp(u)$	Liouville equation
$\exp(u) + \exp(-2u)$	Dodd–Bullough–Mikhailov equation
$\exp(-u) + \exp(-2u)$	Tzitzeica–Dodd–Bullough equation

Moreover, in the Klein–Gordon Eq. (1) the nonlinear term $N(u)$ can be taken as the following polynomial: $N(u) = \gamma_1 u + \gamma_2 u^m$, whenever γ_1 and γ_2 are given constants. In this case, if $m = 2$, Eq. (1) is said to be the Klein-Gordon equation with quadratic nonlinearity, if $m = 3$, the Klein-Gordon equation with cubic nonlinearity, and if $m > 3$ the Klein-Gordon equation with polynomial nonlinearity.

The present investigation focused on the following 2-dimensional nonlinear Klein-Gordon equation [11];

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} + \beta \frac{\partial u(x,y,t)}{\partial t} = \alpha \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) - N(u(x,y,t)) + f(x,y,t), \tag{5}$$

with initial conditions

$$u(x,y,0) = \varphi_1(x,y), \quad \frac{\partial u(x,y,0)}{\partial t} = \varphi_2(x,y), \tag{6}$$

and boundary conditions (Cauchy type BCs)

$$\begin{cases} u(0,y,t) = g_1(y,t), \ \dfrac{\partial}{\partial x} u(0,y,t) = g_2(y,t), \\ u(x,0,t) = g_3(x,t), \ \dfrac{\partial}{\partial y} u(x,0,t) = g_4(y,t), \end{cases} \tag{7}$$

Provided that $(x,y) \in \Omega$ and $t > 0$.

The following is how the remaining part of the paper is organized: Section 2 introduces the 3-dimensional Laplace transform method with some essential definitions, characteristics, and theorems. Section 3 discusses the Daftardar-Gejji and Jafari approach (new iterative method) and its convergence. Section 4, discussed the description of the model; how the solution of the NLKGE is obtained by using the 3-dimensional Laplace transform method linked with the Daftardar-Gejji and Jafari method. In Section 5, the presented method was applied to 3 test examples to show its liability, convergence, and efficiency. Finally, concluding remarks are drawn in Section 6.

Some 3-dimensional Laplace transform method results and theorems

This section provides some basic definitions, properties, and theorems of the 3-dimensional Laplace transform of the partial differential equation, which should be used in the current investigation [22,25,26].

Definition 1: Let $f(t)$ be a function defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $L\{f(t)\}$, is an integral transform given by the Laplace integral;

$$L_t[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt, \tag{8}$$

Provided that this (improper) integral exists, i.e, the integral convergent.

Definition 2: Let $f(x,t)$ be a function of 2 variables x and t defined in the positive $xt-$plane. The double Laplace transform of the function $f(x,t)$ is defined by

$$L_x L_t\left[f(x,t)\right] = F(k,s) = \int_0^\infty\int_0^\infty e^{-kx-st} f(x,t)dtdx, \tag{9}$$

whenever the improper integral exists and k and s are complex numbers.

Definition 3: Let $f(x,y,t)$ be a function of 3 variables $x, y\, and\, t$ defined in the positive $xyt-$plane. The 3-dimensional Laplace transform of the function $f(x,y,t)$ is defined by;

$$L_{xyt}\left[f(x,y,t)\right] = F(k,p,s) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st} f(x,y,t)dtdydx, \tag{10}$$

Provided that the improper integral exists, $L_{xyt}\left[f(x,y,t)\right]$ represents $L_{xyt}\left[f(x,y,t)\right]$, k, p and s are complex numbers and

$$f(x,y,t) = L_{xyt}^{-1}\left[F(k,p,s)\right] = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} e^{-kx}\left(\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{-py}\left[\frac{1}{2\pi i}\int_{z-i\infty}^{z+i\infty} e^{-st}F(k,p,s)ds\right]dp\right)dk, \tag{11}$$

is the inverse 3-dimensional Laplace transform.
From definition above (10), we can conclude that

$$L_{xyt}\left\{f(x)g(y)h(t)\right\} = F(k)G(p)H(s) = L_x\left\{f(x)\right\}L_y\left\{g(y)\right\}L_t\left\{h(t)\right\}. \tag{12}$$

Definition 4: The 3-dimensional Laplace transform of 1st-order partial derivatives of a function of 3 variables $f(x,y,t)$ is given by

$$L_{xyt}\left\{\frac{\partial}{\partial x}f(x,y,t)\right\} = kF(k,p,s) - F(0,p,s),$$
$$L_{xyt}\left\{\frac{\partial}{\partial y}f(x,y,t)\right\} = pF(k,p,s) - F(k,0,s),$$
$$L_{xyt}\left\{\frac{\partial}{\partial t}f(x,y,t)\right\} = sF(k,p,s) - F(k,p,0). \tag{13}$$

Definition 5: The 3-dimensional Laplace transform of 2nd-order partial derivatives of a function of 3 variables $f(x,y,t)$ is given by

$$L_{xyt}\left\{\frac{\partial^2}{\partial x^2}f(x,y,t)\right\} = k^2F(k,p,s) - kF(0,p,s) - \frac{\partial}{\partial x}F(0,p,s),$$
$$L_{xyt}\left\{\frac{\partial^2}{\partial y^2}f(x,y,t)\right\} = p^2F(k,p,s) - pF(k,0,s) - \frac{\partial}{\partial y}F(k,0,s),$$
$$L_{xyt}\left\{\frac{\partial^2}{\partial t^2}f(x,y,t)\right\} = s^2F(k,p,s) - sF(k,p,0) - \frac{\partial}{\partial t}F(k,p,0). \tag{14}$$

Property 6: Assuming that the continuous function $f(x,y,t)$ is triple Laplace transformable, then,

$$L_{xyt}\left[\frac{\partial^3}{\partial x\partial y\partial t}f(x,y,t)\right] = pskF(k,p,s) - kpF(k,p,0) - ksF(k,0,s) + kF(k,0,0)$$
$$- psF(0,p,s) + pF(0,p,0) + sF(0,0,s) - F(0,0,0),$$
$$L_{xxt}\left[\frac{\partial^3}{\partial x^2\partial t}f(x,y,t)\right] = k^2sF(k,p,s) - ksF(0,p,s) - \frac{\partial}{\partial x}F(0,p,s) - s^2F(k,p,0) + sF(0,p,0) + \frac{\partial}{\partial x}F(0,p,0),$$
$$L_{xxx}\left[\frac{\partial^3}{\partial x^3}f(x,y,t)\right] = k^3F(k,p,s) - k^2F(0,p,s) - k\frac{\partial}{\partial x}F(0,p,s) - \frac{\partial^2}{\partial x^2}F(0,p,s). \tag{15}$$

Existence and uniqueness of the 3-dimensional Laplace transform

Theorem 7 (Existence): Let $f(x,y,t)$ be an exponentially ordered continuous function on the interval $[0,\infty]$, that is, for some $a,b,c\in\mathbb{R}$. Consider

$$\sup_{x,y,t>0}\left|\frac{f(x,y,t)}{\exp[ax+by+ct]}\right|<0.$$

Then, the 3-dimensional Laplace transform, $F(k,p,s)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st}f(x,y,t)dtdydx,$ exists for a $p>a,\ s>b\ and\ k>c$ and is in actuality infinitely differentiable with respect to $p>a,\ s>b\ and\ k>c$.

Theorem 8 (Uniqueness): Let $f(x,y,t)$ and $g(x,y,t)$ be continuous functions defined for $x,y,t\geq 0$ and having Laplace transforms, $F(k,p,s)$ and $G(k,p,s)$, respectively. If $F(k,p,s)=G(k,p,s)$, then $f(x,y,t)=g(x,y,t)$.

Some properties of 3-dimensional Laplace transform

This subsection presents the properties of 3-dimensional Laplace transforms that will be used in the current study [22-25].

Property 9 (Linearity): If $f(x,y,t)$ and $g(x,y,t)$ are 2 functions of 3 variables x,y and t such that $L_{xyt}\left[f(x,y,t)\right]=F(k,p,s)$ and $L_{xyt}\left[g(x,y,t)\right]=G(k,p,s)$, then

$$L_{xyt}\{\alpha f(x,y,t)+\beta g(x,y,t)\}=\alpha L_{xyt}f(x,y,t)+\beta L_{xyt}g(x,y,t)=\alpha F(k,p,s)+\beta G(k,p,s), \tag{16}$$

Provided that α and β are constants.

Property 10 (Change of scale property):

If $L_{xyt}\{f(x,y,t)\}=F(k,p,s)$, then $L_{xyt}\{f(ax,by,ct)\}=\frac{1}{abc}F\left(\frac{k}{a},\frac{p}{b},\frac{s}{c}\right)$, (17)

whenever a,b and c are non-zero constants.

Property 11 (1st shifting property):

If $L_{xyt}\{f(x,y,t)\}=F(k,p,s)$, then $L_{xyt}\{e^{ax+by+ct}f(x,y,t)\}=F(k-a,p-b,s-c)$. (18)

Property 12 (2nd Shifting property).

If $L_{xyt}\{f(x,y,t)\}=F(k,p,s)$, then

$$L_{xyt}\{f(x-a,y-b,t-c)H(x-a,y-b,t-c)\}=e^{-ak-bp-cs}F(k,p,s). \tag{19}$$

Remark 13: The function $H(x,y,t)$ in the Eq. (19) is the Heaviside unit step function and is defined by

$$H(x,y,t)=\begin{cases}H(x-a,y-b,t-c);\ x>a,y>b,t>c\\ H(x-a,y-b,t-c);\ x<a,\ \ y<b,\ \ t<c\end{cases}. \tag{20}$$

Property 14: Let $L_{xyt}\{f(x,y,t)\}=F(k,p,s)$, then for all x,y and t,

$$L_{xyt}\left\{e^{ax+by+ct}\right\}=\frac{1}{(k-a)(p-b)(s-c)}. \tag{21}$$

Remark 15: Let $L_{xyt}\{f(x,y,t)\}=F(k,p,s)$, then for all x,y and t, the 3-dimensional Laplace transforms for some function of 3 variables are given as follows [24,25,42]:

i. $L_{xyt}\{abc\}=\frac{abc}{kps}$,
ii. $L_{xyt}\{xyt\}=\frac{1}{k^2p^2s^2}$,

iii. $L_{xyt}\{x^m y^n t^r\} = \frac{m!n!r!}{k^{m+1}p^{n+1}s^{r+1}}$,

iv. $L_{xyt}\{e^{-ax-by-ct}\} = \frac{1}{(a+k)(b+p)(c+s)}$,

v. $L_{xyt}\{cos(x)cos(y)cos(t)\} = \frac{kps}{(1+k^2)(1+p^2)(1+s^2)}$,

vi. $L_{xyt}\{sin(x)sin(y)sin(t)\} = \frac{1}{(1+k^2)(1+p^2)(1+s^2)}$,

vii. $L_{xyt}\{sin(x+y+t)\} = \frac{-1+ps+k(p+s)}{(1+k^2)(1+p^2)(1+s^2)}$,

viii. $L_{xyt}\{cos(x+y+t)\} = -\frac{k+p+s-kps}{(1+k^2)(1+p^2)(1+s^2)}$,

ix. $L_{xyt}\{sinh(ax+by+ct)\} = \frac{1}{2}\left[\frac{1}{(k-a)(p-b)(s-c)} - \frac{1}{(k+a)(p+b)(s+c)}\right]$,

x. $L_{xyt}\{cosh(ax+by+ct)\} = \frac{1}{2}\left[\frac{1}{(k-a)(p-b)(s-c)} + \frac{1}{(k+a)(p+b)(s+c)}\right]$.

Basic idea of the new iterative method (DJM or NIM)

For the basic idea of the new iterative method, we consider the following general functional equation [36]:

$$u(x) = N(u(x)) + f(x), \tag{22}$$

where N is a nonlinear operator in a Banach space such that $N : B \rightarrow B$, and f is a known function. We are looking for a solution u of equation (22) having the series form:

$$u(x) = \sum_{i=0}^{\infty} u_i(x). \tag{23}$$

The nonlinear operation N can then be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty} u_i \left\{N\left(\sum_{r=0}^{i} u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right\}. \tag{24}$$

From Eqs. (24) and (23), Eq. (22) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty} u_i \left\{N\left(\sum_{r=0}^{i} u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right\}. \tag{25}$$

The following recurrence relation is derived from Eq. (25);

$$u_0 = f, \tag{26}$$

$$u_1 = N(u_0), \tag{27}$$

$$u_{(n+1)} = N(u_0 + \cdots + u_n) - N(u_0 + \cdots + u_{(n-1)}),\ n = 1, 2, ... \tag{28}$$

Then,

$$(u_1 + \cdots + u_{(n+1)}) = N(u_0 + \cdots + u_{(n)}),\ n = 1, 2, ..., \tag{29}$$

and hence,

$$u = \sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=1}^{\infty} u_i\right). \tag{30}$$

As a result, the n^{th} approximation series solution of the Eq. (23) is defined as follows:

$$u = u_0 + \sum_{i=0}^{n-1} u_{i+1} = u_0 + u_1 + u_2 + ... + u_{n-1},\ n > 1. \tag{31}$$

Convergence of the DJM

This subsection presents the conditions for the convergence of the series (23). For more details, interested readers may refer to [24,39].

Theorem 16.1: If N is a continuously differentiable functional in a neighborhood of u_0 and $\left\|N^{(n)}(u_0)\right\| \le L,$ for each n and for some real $L>0$ and $\|u_i\| \le M < \frac{1}{e}, i=1,2,3,...,$ then the series $\sum_{i=0}^{\infty} u_{i+1}$ is absolutely convergent and moreover, $\|u_{i+1}\| \le LM^n e^{n-1}(e-1),\ n=1,2,....$

Theorem 16.2: If N is a continuously differentiable functional in a neighborhood of u_0 and $\left\|N^{(n)}(u_0)\right\| \le M \le \frac{1}{e}$ for all n, then the series $\sum_{i=0}^{\infty} u_{i+1}$ is absolutely convergent.

Solution of 2-dimensional NLKGE by 3-dimensional Laplace transform iterative method

To solve problem (5) - (7) by using the proposed method first, the source term $f(x,y,t)$ must be decomposed into 2 functions namely $f_1(x,y,t)$ and $f_2(x,y,t)$. The part $f_1(x,y,t)$ with the terms in Eq. (5) is always leads to the simple algebraic expression while applying the 3-dimensional inverse Laplace transform. The part $f_2(x,y,t)$ is combined with the nonlinear term of Eq. (5) to avoid noise terms in the iteration process.

Steps to find the analytical solution of NLKE (5) - (7) using the proposed techniques are described as follows:

Step 1: Using definition (3) and the 3-dimensional Laplace transform on both sides of the Eq. (5), we get

$$s^2\bar{u}(k,p,s) - s\bar{u}(k,p,0) - \bar{u}_t(k,p,0) + \beta\big(s\bar{u}(k,p,s) - \bar{u}(k,p,0)\big) + \alpha\Big(-p^2\bar{u}(k,p,s) + p\bar{u}(k,0,s) + \bar{u}_y(k,0,s) - k^2\bar{u}(k,p,s) + k\bar{u}(0,p,s) + \bar{u}_x(0,p,s)\Big) + L_{xyt}\Big(N\big(u(x,y,t)\big)\Big) = \overline{f_1}(k,p,s) + L_{xyt}\big(f_2(x,y,t)\big), \tag{32}$$

where $\bar{u}, \bar{u}_t, \bar{u}_x$, $\bar{u}_y$ and $\overline{f_1}$ are Laplace transforms of functions of 3 variables.

Step 2: Applying the 2-dimensional (double) Laplace transform to the Eqs. (2) and (3), we get

$$\bar{u}(k,p,0) = \overline{\varphi_1}(k,p),\ \frac{\partial}{\partial t}\bar{u}(k,p,0) = \overline{\varphi_2}(k,p), \tag{33}$$

$$\begin{cases} \bar{u}(0,p,s) = \overline{g_1}(p,s),\ \ \frac{\partial}{\partial x}\bar{u}(0,p,s) = \overline{g_2}(p,s), \\ \bar{u}(k,0,s) = \overline{g_3}(k,s),\ \ \frac{\partial}{\partial y}\bar{u}(k,0,s) = \overline{g_4}(k,s), \end{cases} \tag{34}$$

where, $\bar{\varphi}_i$, $i = 1,2$; $\bar{g}_i$, $i=1,2,3,4$ are Laplace transforms of functions of 2 variables.

Step 3: Substituting Eqs. (33) and (34) into the Eq. (32) and simplifying, we obtain

$$\bar{u}(k,p,s) = \frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}\begin{Bmatrix} (s+\beta)\bar{u}(k,p,0) + \bar{u}_t(k,p,0) \\ -\alpha\Big(p\bar{u}(k,0,s) + \bar{u}_y(k,0,s) + k\bar{u}(0,p,s) + \bar{u}_x(0,p,s)\Big) \\ +\overline{f_1}(k,p,s) + L_{xyt}\big\{\big(f_2(x,y,t) - N\big(u(x,y,t)\big)\big\} \end{Bmatrix}. \tag{35}$$

Step 4: Applying inverse 3-dimensional Laplace transform to the Eq. (35), we get

$$u(x,y,t) = L_{xyt}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}\begin{Bmatrix} (s+\beta)\bar{u}(k,p,0) + \bar{u}_t(k,p,0) \\ -\alpha\Big(p\bar{u}(k,0,s) + \bar{u}_y(k,0,s) + k\bar{u}(0,p,s) + \bar{u}_x(0,p,s)\Big) \\ +\overline{f_1}(k,p,s) + L_{xyt}\big\{\big(f_2(x,y,t) - N\big(u(x,y,t)\big)\big\} \end{Bmatrix}\right]. \tag{36}$$

Step 5: Applying the iterative process to Eq. (36), assume that the given NLKGE (5) has the series solution of the form:

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t). \tag{37}$$

Step 6: Plugging Eq. (37) into Eq. (36), we obtain

$$\sum_{i=0}^{\infty} u_i(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}\begin{Bmatrix}(s+\beta)\bar{u}(k,p,0)+\bar{u}_t(k,p,0)\\ -\alpha\left(p\bar{u}(k,0,s)+\bar{u}_y(k,0,s)+k\bar{u}(0,p,s)+\bar{u}_x(0,p,s)\right)\\ +\overline{f_1}(k,p,s)+L_{xyt}\{(f_2(x,y,t)-N(u(x,y,t))\}\end{Bmatrix}\right]. \tag{38}$$

Step 7: Decomposing the nonlinear term $N(u(x,y,t))$ in the Eq. (38) as follows;

$$N(\textstyle\sum_{i=0}^{\infty} u_i(x,y,t)) = N(u_0(x,y,t)) + \sum_{i=1}^{\infty}\{N(\sum_{r=0}^{i} u_r(x,y,t)) - N(\sum_{r=0}^{i-1} u_r(x,y,t))\}. \tag{39}$$

Step 8: Using Eq. (39) in the Eq. (38), we get

$$\sum_{i=0}^{\infty} u_i(x,y,t)\, L_{xyt}{}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}\begin{Bmatrix}\overline{f_1}(k,p,s)+(s+\beta)\bar{u}(k,p,0)+\bar{u}_t(k,p,0)\\ -\alpha\left(p\bar{u}(k,0,s)+\bar{u}_y(k,0,s)+k\bar{u}(0,p,s)+\bar{u}_x(0,p,s)\right)\\ +L_{xyt}\{(f_2(x,y,t)-N(u_0(x,y,t))\}\\ +L_{xyt}\{\sum_{i=1}^{\infty}\{N\sum_{r=0}^{i} u_r(x,y,t)-N[\sum_{r=0}^{i-1} u_r(x,y,t)]\}\}\end{Bmatrix}\right]. \tag{40}$$

Step 9: Define the following recurrence relation from the equation in the sense of (25);

$$u_0(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}\begin{Bmatrix}\overline{f_1}(k,p,s)+(s+\beta)\bar{u}(k,p,0)+\bar{u}_t(k,p,0)\\ -\alpha\left(p\bar{u}(k,0,s)+\bar{u}_y(k,0,s)+k\bar{u}(0,p,s)+\bar{u}_x(0,p,s)\right)\end{Bmatrix}\right], \tag{41}$$

$$u_1(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}L_{xyt}\{(f_2(x,y,t)-N(u_0(x,y,t))\}\right], \tag{42}$$

$$u_{n+1}u(x,y,t) = -L_{xyt}{}^{-1}\left[\frac{1}{s^2-\alpha p^2-\alpha k^2+\beta s}L_{xyt}\{\textstyle\sum_{i=1}^{\infty}\{N\sum_{r=0}^{i} u_r(x,y,t) - N[\sum_{r=0}^{i-1} u_r(x,y,t)]\}\}\right], n \geq 1. \tag{43}$$

Step 10: Using (31) and write the series solution of NKG Eqs. (5) - (7) is as follows:

$$u(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+...+u_n(x,y,t)+.... \tag{44}$$

Step 11: Determine the convergence of the series solution (44) by applying the stated theorem 16.

Applications

In this section, certain test examples will be studied to evaluate the suggested method's performance. To verify the convergence of the new iterative (Daftardar-Gejji and Jafari) method, theorem 16 will be applied to each considered illustrative problem.

Example 1: Consider the following (2+1)-dimensional NLKGE with quadratic nonlinearity [11].

That is;

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + u^2(x,y,t) = -xy\cos t + x^2y^2\cos^2 t,\ t>0, \tag{45}$$

with initial conditions

$$u(x,y,0) = xy,\ \frac{\partial u(x,y,0)}{\partial t} = 0, \tag{46}$$

and boundary conditions

$$\begin{cases} u(0,y,t) = 0,\ \frac{\partial}{\partial x}u(0,y,t) = ycost, \\ u(x,0,t) = 0,\ \frac{\partial}{\partial y}u(x,0,t) = xcost. \end{cases} \tag{47}$$

Solution: Applying the properties of 3-dimensional Laplace transform on both sides of the Eq. (45), we get

$$\left\{\begin{array}{l} s^2\bar{u}(k,p,s) - s\bar{u}(k,p,0) - \frac{\partial}{\partial t}\bar{u}(k,p,0) - p^2\bar{u}(k,p,s) + p\bar{u}(k,0,s) \\ +\frac{\partial}{\partial y}\bar{u}(k,0,s) - k^2\bar{u}(k,p,s) + k\bar{u}(0,p,s) + \frac{\partial}{\partial x}\bar{u}(0,p,s) + L_{xyt}\big(u^2(x,y,t)\big) \end{array}\right\}$$
$$= -\frac{s}{k^2p^2(s^2+1)} + L_{xyt}[x^2y^2cos^2t]. \tag{48}$$

Applying the 2-dimensional (double) Laplace transform to Eqs. (46) and (47), we obtain

$$\bar{u}(k,p,0) = \frac{1}{k^2p^2},\ \ \frac{\partial}{\partial t}\bar{u}(k,p,0) = 0, \tag{49}$$

$$\begin{cases} \bar{u}(0,p,s) = 0, & \frac{\partial}{\partial x}\bar{u}(0,p,s) = \frac{s}{p^2(s^2+1)}, \\ \bar{u}(k,0,s) = 0, & \frac{\partial}{\partial y}\bar{u}(k,0,s) = \frac{s}{k^2(s^2+1)}. \end{cases} \tag{50}$$

Substituting Eqs. (49) and (50) into the Eq. (48), we get

$$\bar{u}(k,p,s) = \frac{1}{(s^2-p^2-k^2)}\Big(-\frac{s}{k^2p^2(s^2+1)} + \frac{s}{k^2p^2} - \frac{s}{k^2(s^2+1)} - \frac{s}{p^2(s^2+1)} + L_{xyt}[x^2y^2cos^2t - u^2(x,y,t)]\Big). \tag{51}$$

By simplifying the Eq. (51), we obtain

$$\bar{u}(k,p,s) = \frac{s}{k^2p^2(s^2+1)} + \frac{1}{(s^2-p^2-k^2)}\ L_{xyt}[x^2y^2cos^2t - u^2(x,y,t)]. \tag{52}$$

Applying the inverse 3-dimensional Laplace transform to the equation (52), we get

$$u(x,y,t) = xycost + L_{xyt}{}^{-1}\Big\{\frac{1}{(s^2-p^2-k^2)}\ L_{xyt}[x^2y^2cos^2t -\ u^2(x,y,t)]\Big\}. \tag{53}$$

Now, apply the new Iterative method.
Substituting (39) into (53) and applying (41), (42), and (43), we obtain the components of the solution as follows:

$$u_0(x,y,t) = xycost, \tag{54}$$

$$u_1(x,y,t) = L_{xyt}{}^{-1}\Big\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[x^2y^2cos^2t - x^2y^2cos^2t]\Big\} = 0. \tag{55}$$

$$u_{n+1}u(x,y,t) = -\,L_{xyt}{}^{-1}\Big[\frac{1}{s^2-p^2-k^2}\ L_{xyt}\{\textstyle\sum_{r=0}^{n} u_r{}^2(x,y,t) - \sum_{r=0}^{n-1} u_r{}^2(x,y,t)\}\Big], n \geq 1. \tag{56}$$

Now, we define the recurrence relation from an equation (56) for $n \geq 1$ as follows:

$$u_2(x,y,t) = L_{xyt}{}^{-1}\Big[\frac{1}{(s^2-p^2-k^2)}L_{xyt}\Big\{\big(u_0(x,y,t) + u_1(x,y,t)\big)^2 -\ u_0{}^2(x,y,t)\Big\}\Big] = 0, \tag{57}$$

$$u_3(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left\{\begin{array}{l}\big(u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t)\big)^2 \\ -\big(u_0(x,y,t) + u_1(x,y,t)\big)^2\end{array}\right\}\right] = 0. \tag{58}$$

Similarly, we get $u_4(x,y,t) = u_5(x,y,t) = 0$ and so on.
Therefore, the solution of Example 1 by using Eq. (44) is

$$u(x,y,t) = xy\cos t. \tag{59}$$

Now, theorem 15 is applied as follows to show that the resulting series solution is convergent.
From the Eq. (53), we have
$u_0(x,y,t) = xycost$ and

$$N\big(u(x,y,t)\big) = L_{xyt}{}^{-1}\Big\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[x^2y^2cos^2t - u^2(x,y,t)]\Big\}. \tag{60}$$

Hence for all $x,y,t \geq 0$ from Eq. (60), we obtain

$$N\big(u_0(x,y,t)\big) = L_{ytx}{}^{-1}\Big\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[x^2y^2cos^2t - u_0{}^2(x,y,t)]\Big\}$$
$$= L_{xyt}{}^{-1}\Big\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[x^2y^2cos^2t - x^2y^2cos^2t]\Big\} = 0,$$

which implies that, $\left\|N\left(u_0(x,y,t)\right)\right\| = \|0\| = 0 < \frac{1}{e}$.

$N'\left(u(x,y,t)\right) = L_{ytx}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[2xycost(xycost)' - 2u(x,y,t)u'(x,y,t)]\right\}$,

where $N'\left(u(x,y,t)\right)$ represents the partial derivatives $\frac{\partial}{\partial x}u(x,y,t)$ or $\frac{\partial}{\partial y}u(x,y,t)$ or $\frac{\partial}{\partial t}u(x,y,t)$.

$$N\left(\frac{\partial}{\partial t}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[-2xycost(xysint) - 2u(x,y,t)\frac{\partial}{\partial t}u(x,y,t)\right]\right\}. \quad (61)$$

From Eq. (61), we get

$$\begin{aligned} N\left(\frac{\partial}{\partial t}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[-2xycost(xysint) - 2u_0(x,y,t)\frac{\partial}{\partial t}u_0(x,y,t)\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[-2xycost(xysint) + 2xycost(xysint)]\right\} = 0. \end{aligned}$$

$$N\left(\frac{\partial}{\partial x}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[2xy^2cos^2t - 2u(x,y,t)\frac{\partial}{\partial x}u(x,y,t)\right]\right\}. \quad (62)$$

From Eq. (62), we obtain

$$\begin{aligned} N\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[2xy^2cos^2t - 2u_0(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t)\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[2xy^2cos^2t - 2xy^2cos^2t]\right\} = 0. \end{aligned}$$

$$N\left(\frac{\partial}{\partial y}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[2x^2ycos^2t - 2u(x,y,t)\frac{\partial}{\partial y}u(x,y,t)\right]\right\}. \quad (63)$$

From Eq. (63), we get

$$\begin{aligned} N\left(\frac{\partial}{\partial y}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[2x^2ycos^2t - 2u_0(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t)\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[2x^2ycos^2t - 2x^2ycos^2t]\right\} = 0. \end{aligned}$$

Therefore, $\left\|N\left(u_0{}'(x,y,t)\right)\right\| = \|0\| = 0 < \frac{1}{e}$.

$$\begin{aligned} N\left(\frac{\partial^2}{\partial t^2}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{-2}{(s^2-p^2-k^2)} L_{xyt}\left[\begin{array}{l} -(xysint)^2 + (xycost)^2 \\ +\left(\frac{\partial}{\partial t}u_0(x,y,t)\right)^2 + u_0(x,y,t)\frac{\partial^2}{\partial t^2}u_0(x,y,t) \end{array}\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{-2}{(s^2-p^2-k^2)} L_{xyt}[-(xysint)^2 + (xycost)^2 + (xysint)^2 - (xycost)^2]\right\} = 0. \end{aligned} \quad (64)$$

$$\begin{aligned} N\left(\frac{\partial^2}{\partial x^2}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{2}{(s^2-p^2-k^2)} L_{xyt}\left[y^2cos^2t - \left(\left(\frac{\partial}{\partial x}u_0(x,y,t)\right)^2 + u_0(x,y,t)\frac{\partial^2}{\partial x^2}u_0(x,y,t)\right)\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{-2}{(s^2-p^2-k^2)} L_{xyt}[y^2cos^2t - y^2cos^2t]\right\} = 0. \end{aligned} \quad (65)$$

$$\begin{aligned} N\left(\frac{\partial^2}{\partial y^2}u_0(x,y,t)\right) &= L_{xyt}{}^{-1}\left\{\frac{2}{(s^2-p^2-k^2)} L_{xyt}\left[x^2cos^2t - \left(\left(\frac{\partial}{\partial y}u_0(x,y,t)\right)^2 + u_0(x,y,t)\frac{\partial^2}{\partial y^2}u_0(x,y,t)\right)\right]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{-2}{(s^2-p^2-k^2)} L_{xyt}[x^2cos^2t - x^2cos^2t]\right\} = 0. \end{aligned} \quad (66)$$

$$N\left(\frac{\partial^2}{\partial x\partial y}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[4xycos^2t - 4xycos^2t]\right\} = 0. \quad (67)$$

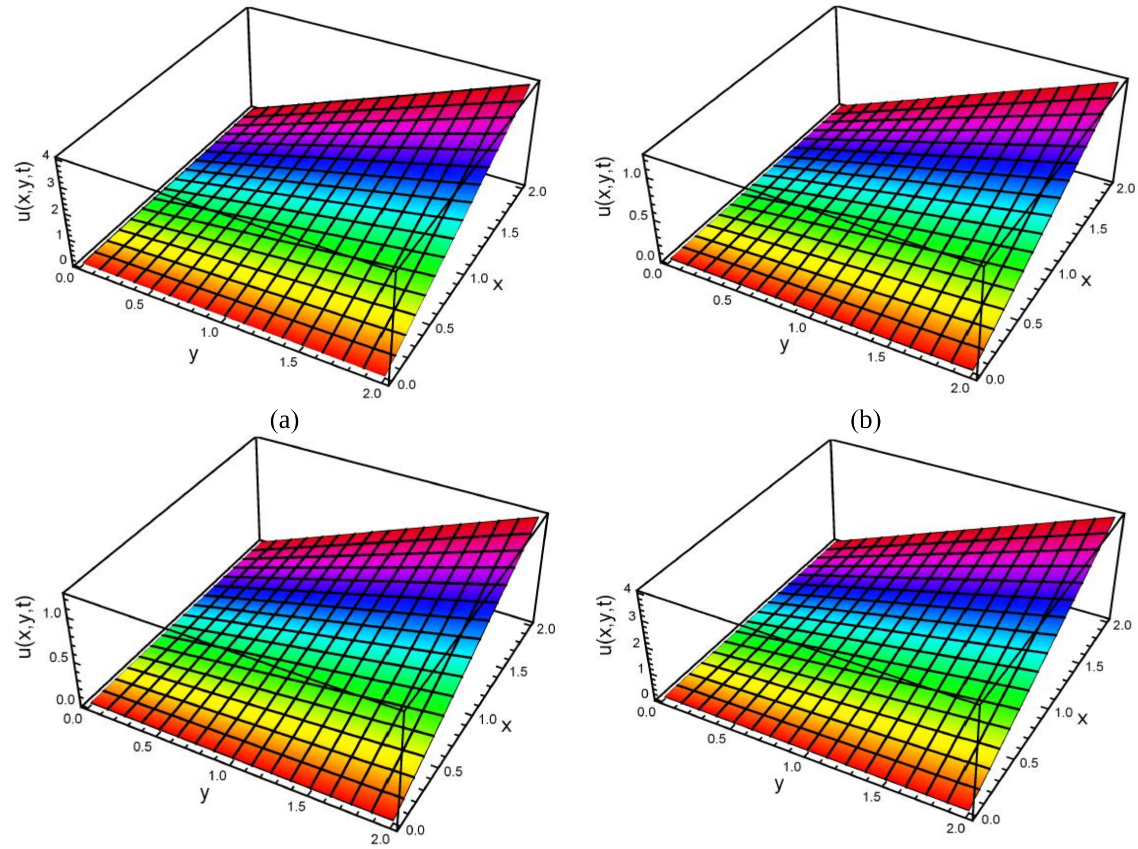
$$N\left(\frac{\partial^2}{\partial x\partial t}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[-4xy^2costsint + 4xy^2costsint]\right\} = 0. \quad (68)$$

$$N\left(\frac{\partial^2}{\partial y\partial t}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[-4x^2ycostsint + 4x^2ycostsint]\right\} = 0. \quad (69)$$

Therefore, $\left\|N\left(u_0{}''(x,y,t)\right)\right\| = \|0\| = 0 < \frac{1}{e}$.

Using the same procedure, we obtain $\left\|N^{(3)}\left(u_0(x,y,t)\right)\right\| = \left\|N^{(4)}\left(u_0(x,y,t)\right)\right\| = \cdots = \left\|N^{(k)}\left(u_0(x,y,t)\right)\right\| = 0 < \frac{1}{e}$, for all $k \geq 0$.

The series solution $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$ obtained by Daftardar-Gejji and Jafari (New iterative) approach is completely convergent on the domain of interest since the criteria of Theorem 16 are satisfied.



(a) (b) (c) (d)

Figure 1 3D solution plots of Example 1 at a) t = 0, b) t = 0.5π c) t = 1.5π, and d) t = 2π.

Example 2: Consider the following (2+1)-dimensional NLKGE with cubic nonlinearity on the domain $(x,y)=\left[0,\,\pi\right]^2,\ t\geq 0$ [7,11];

$$\frac{\partial^2 u(x,y,t)}{\partial t^2}-\frac{\partial^2 u(x,y,t)}{\partial x^2}-\frac{\partial^2 u(x,y,t)}{\partial y^2}+u^3(x,y,t)=\cos x\cos y\sin t+\cos^3 x\cos^3 y\sin^3 t, \tag{70}$$

with initial conditions

$$u(x,y,0)=0,\ \frac{\partial u(x,y,0)}{\partial t}=\cos x\cos y, \tag{71}$$

and boundary conditions

$$\begin{cases} u(0,y,t) = cosysint,\ \frac{\partial}{\partial x}u(0,y,t) = 0, \\ u(x,0,t) = cosxsint,\ \frac{\partial}{\partial y}u(x,0,t) = 0. \end{cases} \tag{72}$$

Solution: Applying the properties of 3-dimensional Laplace transform on both sides of the Eq. (70) , we get

$$\left\{\begin{array}{c} s^2\bar{u}(k,p,s) - s\bar{u}(k,p,0) - \dfrac{\partial}{\partial t}\bar{u}(k,p,0) - p^2\bar{u}(k,p,s) + p\bar{u}(k,0,s) \\ +\dfrac{\partial}{\partial y}\bar{u}(k,0,s) - k^2\bar{u}(k,p,s) + k\bar{u}(0,p,s) + \dfrac{\partial}{\partial x}\bar{u}(0,p,s) + L_{xyt}\left(u^3(x,y,t)\right) \end{array}\right\}$$
$$= \frac{kp}{(1+k^2)(1+p^2)(1+s^2)} + L_{xyt}[cos^3xcos^3ysin^3t]. \tag{73}$$

Taking the 2-dimensional (double) Laplace transform to Eqs. (71) and (72), we obtain

$$\bar{u}(k,p,0) = 0,\ \ \frac{\partial}{\partial t}\bar{u}(k,p,0) = \frac{kp}{(1+k^2)(1+p^2)}, \tag{74}$$

$$\begin{cases} \bar{u}(0,p,s) = \frac{p}{(1+s^2)(1+p^2)}, & \frac{\partial}{\partial x}\bar{u}(0,p,s) = 0, \\ \bar{u}(k,0,s) = \frac{k}{(1+s^2)(1+k^2)}, & \frac{\partial}{\partial y}\bar{u}(k,0,s) = 0. \end{cases} \tag{75}$$

Substituting Eqs. (74) and (75) into the Eq. (73), we get

$$\bar{u}(k,p,s) = \frac{1}{(s^2-p^2-k^2)}\left(\begin{array}{c} \frac{kp}{(1+k^2)(1+p^2)(1+s^2)} + \frac{kp}{(1+k^2)(1+p^2)} - \frac{kp}{(1+s^2)(1+k^2)} \\ -\frac{kp}{(1+s^2)(1+p^2)} + L_{xyt}[cos^3xcos^3ysin^3t - u^3(x,y,t)] \end{array} \right). \tag{76}$$

By simplifying the Eq. (76) , we obtain

$$\bar{u}(k,p,s) = \frac{kp}{(1+k^2)(1+p^2)(1+s^2)} + \frac{1}{(s^2-p^2-k^2)}\ L_{xyt}[cos^3xcos^3ysin^3t - u^3(x,y,t)]. \tag{77}$$

Taking the inverse 3-dimensional Laplace transform on both sides of the Eq. (77) , we get

$$u(x,y,t) = \cos x \cos y \sin t + L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}\ L_{xyt}[cos^3xcos^3ysin^3t - u^3(x,y,t)]\right\}. \tag{78}$$

Now, apply the new Iterative method.
Using (39) into (78) and applying (41), (42), and (43), we obtain the components of the solution as follows:

$$u_0(x,y,t) = \cos x \cos y \sin t\,, \tag{79}$$

$$u_1(x,y,t) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[cos^3xcos^3ysin^3t - cos^3xcos^3ysin^3t]\right\} = 0. \tag{80}$$

$$u_{n+1}u(x,y,t) = -L_{xyt}{}^{-1}\left[\frac{1}{s^2-p^2-k^2}\ L_{xyt}\{\textstyle\sum_{r=0}^{n} u_r{}^3(x,y,t) - \sum_{r=0}^{n-1} u_r{}^3(x,y,t)\}\right], n \geq 1. \tag{81}$$

Now, we define the recurrence relation from the equation (81) for $n \geq 1$ as follows:

$$u_2(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left\{\left(u_0(x,y,t) + u_1(x,y,t)\right)^3 - u_0{}^3(x,y,t)\right\}\right] = 0, \tag{82}$$

$$u_3(x,y,t) = L_{xyt}{}^{-1}\left[\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left\{ \begin{array}{l} \left(u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t)\right)^3 \\ -\left(u_0(x,y,t) + u_1(x,y,t)\right)^3 \end{array} \right\}\right] = 0. \tag{83}$$

In the same way above procedure, we obtain
$u_4(x,y,t) = u_5(x,y,t) = 0$ and so on.
As a result, the solution of Example 2 in view of (44) is

$$u(x,y,t) = \cos x \cos y \sin t\,. \tag{84}$$

Now, theorem 15 is applied as follows to show that the resulting series solution is convergent.
From Eq. (81) , we have

$u_0(x,y,t) = \cos x \cos y \sin t$ and

$$N\big(u(x,y,t)\big) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[cos^3xcos^3ysin^3t - u^3(x,y,t)]\right\}. \tag{85}$$

Thus, for all $x, y, t \geq 0$ from Eq. (85), we obtain

$$\begin{aligned} N\big(u_0(x,y,t)\big) &= L_{ytx}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[cos^3xcos^3ysin^3t - u_0{}^3(x,y,t)]\right\} \\ &= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}[cos^3xcos^3ysin^3t - cos^3xcos^3ysin^3t]\right\} = 0, \end{aligned}$$

Hence, $\left\|N\big(u_0(x,y,t)\big)\right\| = \|0\| = 0 < \frac{1}{e}$.

$$N'\big(u(x,y,t)\big) = L_{ytx}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)} L_{xyt}\left[\begin{array}{c} 3\left(\cos x \cos y \sin t\right)^2 \left(\cos x \cos y \sin t\right)' \\ -3u^2(x,y,t)u'(x,y,t) \end{array} \right]\right\}, \tag{86}$$

where $N'\left(u(x,y,t)\right)$ represents the partial derivatives $\frac{\partial}{\partial x}u(x,y,t)$ or $\frac{\partial}{\partial y}u(x,y,t)$ or $\frac{\partial}{\partial t}u(x,y,t)$.

$$N\left(\frac{\partial}{\partial t}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[3cos^3xcos^3ysin^3t - 3u^2(x,y,t)\frac{\partial}{\partial t}u(x,y,t)\right]\right\}. \tag{87}$$

From the Eq. (87), we get

$$N\left(\frac{\partial}{\partial t}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[3cos^3xcos^3ysin^3t - 3u_0{}^2(x,y,t)\frac{\partial}{\partial t}u_0(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[3cos^3xcos^3ysin^3t - 3cos^3xcos^3ysin^3t]\right\} = 0.$$

$$N\left(\frac{\partial}{\partial x}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3\cos^2 x\cos^3 y\sin^3 t\sin x - 3u^2(x,y,t)\frac{\partial}{\partial x}u(x,y,t)\right]\right\}. \tag{88}$$

From Eq. (88), we obtain

$$N\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3\cos^2 x\cos^3 y\sin^3 t\sin x - 3u_0{}^2(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3\cos^2 x\cos^3 y\sin^3 t\sin x + 3\cos^2 x\cos^3 y\sin^3 t\sin x\right]\right\} = 0.$$

$$N\left(\frac{\partial}{\partial y}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3u^2(x,y,t)\frac{\partial}{\partial y}u(x,y,t) - 3\cos^3 x\cos^2 y\sin^3 t\sin y\right]\right\}. \tag{89}$$

From Eq. (89), we get

$$N\left(\frac{\partial}{\partial y}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3\cos^3 x\cos^2 y\sin^3 t\sin y - 3u_0{}^2(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-3\cos^3 x\cos^2 y\sin^3 t\sin y + 3\cos^3 x\cos^2 y\sin^3 t\sin y\right]\right\} = 0.$$

Hence, $\left\|N\left(u_0{}'(x,y,t)\right)\right\| = \|0\| = 0 < \frac{1}{e}$.

$$N\left(\frac{\partial^2}{\partial t^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 x\cos^3 y\left[2\sin t\cos^2 t - \sin^3 t\right]\\ +\left(2u(x,y,t)\left(\frac{\partial}{\partial t}u_0(x,y,t)\right)^2 + u_0{}^2(x,y,t)\frac{\partial^2}{\partial t^2}u_0(x,y,t)\right)\end{array}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 x\cos^3 y\left[2\sin t\cos^2 t - \sin^3 t\right]\\ -\cos^3 x\cos^3 y\left[2\sin t\cos^2 t - \sin^3 t\right]\end{array}\right]\right\} = 0. \tag{90}$$

$$N\left(\frac{\partial^2}{\partial x^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 y\cos^3 t\left[-2\cos x\sin^2 x + \cos^3 x\right]\\ +\left(\begin{array}{c}2u(x,y,t)\left(\frac{\partial}{\partial x}u_0(x,y,t)\right)^2\\ +u_0{}^2(x,y,t)\frac{\partial^2}{\partial x^2}u_0(x,y,t)\end{array}\right)\end{array}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 y\cos^3 t\left[-2\cos x\sin^2 x + \cos^3 x\right]\\ -\cos^3 y\cos^3 t\left[-2\cos x\sin^2 x + \cos^3 x\right]\end{array}\right]\right\} = 0. \tag{91}$$

$$N\left(\frac{\partial^2}{\partial y^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 x\cos^3 t\left[-2\cos y\sin^2 y + \cos^3 y\right]\\ +\left(\begin{array}{c}2u(x,y,t)\left(\frac{\partial}{\partial y}u_0(x,y,t)\right)^2\\ +u_0{}^2(x,y,t)\frac{\partial^2}{\partial y^2}u_0(x,y,t)\end{array}\right)\end{array}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{-3}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}\cos^3 x\cos^3 t\left[-2\cos y\sin^2 y + \cos^3 y\right]\\ -\cos^3 x\cos^3 t\left[-2\cos y\sin^2 y + \cos^3 y\right]\end{array}\right]\right\} = 0. \tag{92}$$

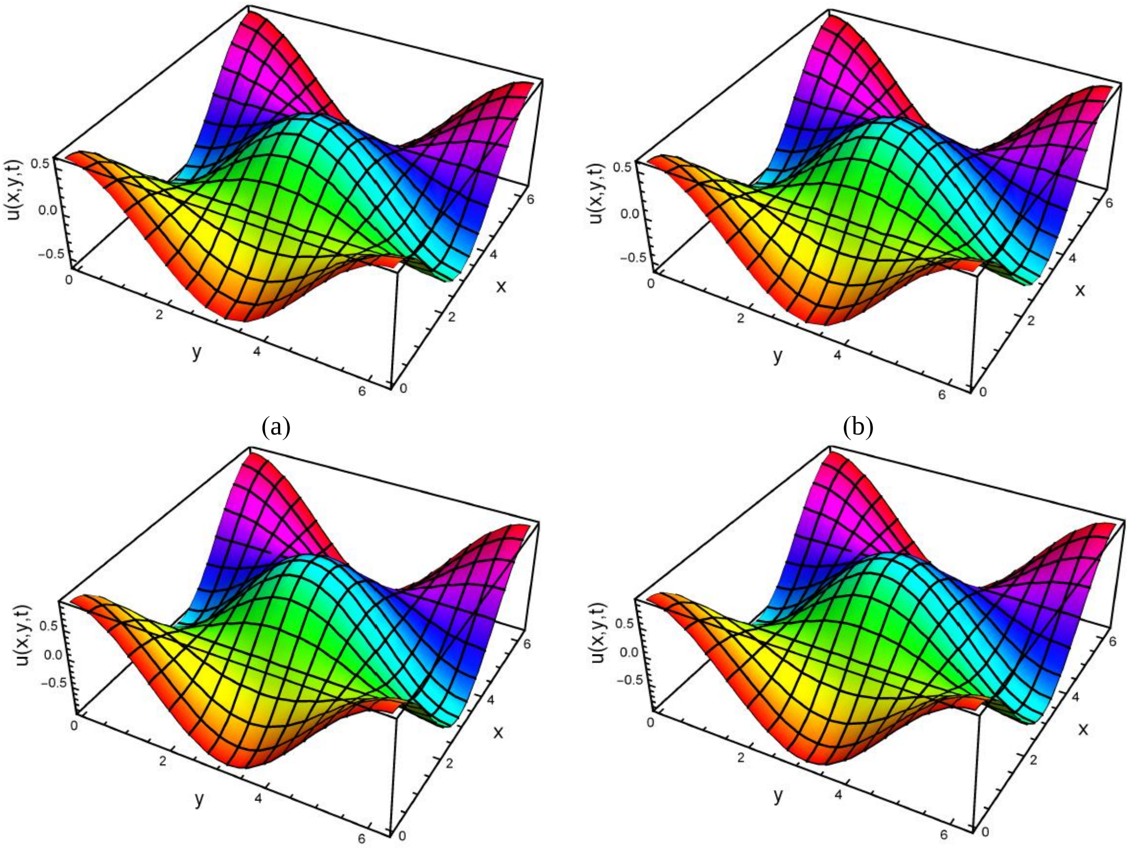
$$N\left(\frac{\partial^2}{\partial x\partial y}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[\begin{array}{c}9\sin^3 t\cos^2 x\cos^2 y\sin x\sin y\\ -9\sin^3 t\cos^2 x\cos^2 y\sin x\sin y\end{array}\right]\right\} = 0. \tag{93}$$

$$N\left(\frac{\partial^2}{\partial x\partial t}u_0(x,y,t)\right) = {L_{xyt}}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\begin{bmatrix}-9\cos^3 y\cos^2 x\sin^2 t\sin x\cos t\\ +9\cos^3 y\cos^2 x\sin^2 t\sin x\cos t\end{bmatrix}\right\} = 0. \tag{94}$$

$$N\left(\frac{\partial^2}{\partial y\partial t}u_0(x,y,t)\right) = {L_{xyt}}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\begin{bmatrix}-9\cos^3 x\cos^2 y\sin^2 t\sin y\cos t\\ +9\cos^3 x\cos^2 y\sin^2 t\sin y\cos t\end{bmatrix}\right\} = 0. \tag{95}$$

Therefore, $\|N(u_0''(x,y,t))\|=\|0\| = 0 < \frac{1}{e}$.
Similarly, we get $\|N^{(3)}(u_0(x,y,t))\| = \|N^{(4)}(u_0(x,y,t))\| = \cdots = \|N^{(k)}(u_0(x,y,t))\| = 0 < \frac{1}{e}$, for all $k \geq 0$.

The series solution $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$ obtained by Daftardar-Gejji and Jafari (New iterative) approach is completely convergent on the domain of interest since the criteria of Theorem 15 are satisfied.



(a) (b)

(c) (d)

Figure 2 3D solution plots of Example 2 at a) $t = 0.2\,\pi$, b) $t = 0.4\pi$ c) $t = 0.6\pi$, and d) $t = 0.8\pi$.

Example 3: Consider the following (2+1)-dimensional NLKGE with polynomial type nonlinearity of degree 5;

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + u^5(x,y,t) = -e^{x+y-t} + e^{5x+5y-5t},\ t \geq 0, \tag{96}$$

with initial conditions

$$u(x,y,0)=e^{x+y},\ \frac{\partial u(x,y,0)}{\partial t}=-e^{x+y}, \tag{97}$$

and boundary conditions

$$\begin{cases} u(0,y,t)=e^{y-t},\ \frac{\partial}{\partial x}u(0,y,t)=e^{y-t}, \\ u(x,0,t)=e^{x-t},\ \frac{\partial}{\partial y}u(x,0,t)=e^{x-t}. \end{cases} \tag{98}$$

Solution: Applying the properties of 3-dimensional Laplace transform on both sides of the Eq. (96), we get

$$\begin{Bmatrix} s^2\bar{u}(k,p,s)-s\bar{u}(k,p,0)-\frac{\partial}{\partial t}\bar{u}(k,p,0)-p^2\bar{u}(k,p,s)+p\bar{u}(k,0,s) \\ +\frac{\partial}{\partial y}\bar{u}(k,0,s)-k^2\bar{u}(k,p,s)+k\bar{u}(0,p,s)+\frac{\partial}{\partial x}\bar{u}(0,p,s)+L_{xyt}\left(u^5(x,y,t)\right) \end{Bmatrix} \\ =-\frac{1}{(1-k)(1-p)(1+s)}+L_{xyt}[e^{5x+5y-5t}]. \tag{99}$$

Applying double Laplace transform to Eqs. (97) and (98), we obtain

$$\bar{u}(k,p,0)=\frac{1}{(1-k)(1-p)},\ \frac{\partial}{\partial t}\bar{u}(k,p,0)=-\frac{1}{(1-k)(1-p)}, \tag{100}$$

$$\begin{cases} \bar{u}(0,p,s)=\frac{1}{(1-p)(1+s)},\ \ \frac{\partial}{\partial x}\bar{u}(0,p,s)=\frac{1}{(1-p)(1+s)}, \\ \bar{u}(k,0,s)=\frac{1}{(1-k)(1+s)},\ \ \frac{\partial}{\partial y}\bar{u}(k,0,s)=\frac{1}{(1-k)(1+s)}. \end{cases} \tag{101}$$

Substituting Eqs. (100) and (101) into Eq. (99), we get

$$\bar{u}(k,p,s)=\frac{1}{(s^2-p^2-k^2)}\begin{pmatrix} -\frac{1}{(1-k)(1-p)(1+s)}+\frac{s-1}{(1-k)(1-p)}-\frac{p+1}{(1-k)(1+s)} \\ -\frac{k+1}{(1-p)(1+s)}+L_{xyt}[e^{5x+5y-5t}-u^5(x,y,t)] \end{pmatrix}. \tag{102}$$

Simplifying (102), gives us

$$\bar{u}(k,p,s)=\frac{1}{(1-k)(1-p)(1+s)}+\frac{1}{(s^2-p^2-k^2)}L_{xyt}[e^{5x+5y-5t}-u^5(x,y,t)]. \tag{103}$$

Applying inverse 3-dimensional Laplace transform to Eq. (103), we get

$$u(x,y,t)=e^{x+y-t}+L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[e^{5x+5y-5t}-u^5(x,y,t)]\right\}. \tag{104}$$

Now, apply the new Iterative method.
Substituting (39) into (104) and applying (41), (42), and (43), we obtain the components of the solution as follows:

$$u_0(x,y,t)=e^{x+y-t}, \tag{105}$$

$$u_1(x,y,t)=L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}[e^{5x+5y-5t}-e^{5x+5y-5t}]\right\}=0. \tag{106}$$

$$u_{n+1}u(x,y,t)=-L_{xyt}{}^{-1}\left[\frac{1}{s^2-p^2-k^2}L_{xyt}\{\textstyle\sum_{r=0}^{n}u_r{}^5(x,y,t)-\sum_{r=0}^{n-1}u_r{}^5(x,y,t)\}\right], n\geq 1. \tag{107}$$

Now, we define the recurrence relation from Eq. (107) for $n\geq 1$ as follows:

$$u_2(x,y,t)=L_{xyt}{}^{-1}\left[\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left\{\left(u_0(x,y,t)+u_1(x,y,t)\right)^5-u_0{}^5(x,y,t)\right\}\right]=0, \tag{108}$$

$$u_3(x,y,t)=L_{xyt}{}^{-1}\left[\frac{1}{(s^2-p^2-k^2)}L_{xyt}\begin{Bmatrix} \left(u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)\right)^5 \\ -\left(u_0(x,y,t)+u_1(x,y,t)\right)^5 \end{Bmatrix}\right]=0. \tag{109}$$

Using the same procedure above, we have $u_4(x,y,t)=u_5(x,y,t)=0$ and so on.
Therefore, the solution of Example 3 by using Eq. (44) is

$$u(x,y,t)=e^{x+y-t}. \tag{110}$$

Now, theorem 15 is applied as follows to show that the resulting series solution is convergent.

From the Eq. (107) , we have
$u_0(x,y,t) = e^{x+y-t}$ and
$$N\big(u(x,y,t)\big) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[e^{5x+5y-5t} - u^5(x,y,t)\right]\right\}. \tag{111}$$

Thus, for all $x, y, t \geq 0$ from Eq. (111), we obtain
$$N\big(u_0(x,y,t)\big) = L_{ytx}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[e^{5x+5y-5t} - u_0{}^5(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[e^{5x+5y-5t} - e^{5x+5y-5t}\right]\right\} = 0,$$
which implies that, $\left\|N\big(u_0(x,y,t)\big)\right\| = \|0\| = 0 < \frac{1}{e}$.
$$N'\big(u(x,y,t)\big) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[e^{5x+5y-5t}\left(5x+5y-5t\right)' - 5u^4(x,y,t)u'(x,y,t)\right]\right\},$$
where $N'\big(u(x,y,t)\big)$ represents the partial derivatives $\frac{\partial}{\partial x}u(x,y,t)$ or $\frac{\partial}{\partial y}u(x,y,t)$ or $\frac{\partial}{\partial t}u(x,y,t)$
$$N\left(\frac{\partial}{\partial t}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-5e^{5x+5y-5t} - 5u^4(x,y,t)\frac{\partial}{\partial t}u(x,y,t)\right]\right\}. \tag{112}$$

From Eq. (112), we get $N\left(\frac{\partial}{\partial t}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-5e^{5x+5y-5t} - 5u_0{}^4(x,y,t)\frac{\partial}{\partial t}u_0(x,y,t)\right]\right\}$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-5e^{5x+5y-5t} + 5e^{5x+5y-5t}\right]\right\} = 0.$$
$$N\left(\frac{\partial}{\partial x}u(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[5e^{5x+5y-5t} - 5u^4(x,y,t)\frac{\partial}{\partial x}u(x,y,t)\right]\right\}. \tag{113}$$

From Eq. (113), we obtain
$$N\left(\frac{\partial}{\partial x}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[5e^{5x+5y-5t} - 5u_0{}^4(x,y,t)\frac{\partial}{\partial x}u_0(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[5e^{5x+5y-5t} - 5e^{5x+5y-5t}\right]\right\} = 0.$$
$$N\left(\frac{\partial}{\partial y}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[5e^{5x+5y-5t} - 5u_0{}^4(x,y,t)\frac{\partial}{\partial y}u_0(x,y,t)\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[5e^{5x+5y-5t} - 5e^{5x+5y-5t}\right]\right\} = 0.$$
Therefore, $\left\|N\big(u_0{}'(x,y,t)\big)\right\| = \|0\| = 0 < \frac{1}{e}$.
$$N\left(\frac{\partial^2}{\partial t^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 5\begin{pmatrix}4u_0{}^3(x,y,t)\left(\frac{\partial}{\partial t}u_0(x,y,t)\right)^2\\ +u_0{}^4(x,y,t)\frac{\partial^2}{\partial t^2}u_0(x,y,t)\end{pmatrix}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 25e^{5x+5y-5t}\right]\right\} = 0. \tag{114}$$

$$N\left(\frac{\partial^2}{\partial x^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 5\begin{pmatrix}4u_0{}^3(x,y,t)\left(\frac{\partial}{\partial x}u_0(x,y,t)\right)^2\\ +u_0{}^4(x,y,t)\frac{\partial^2}{\partial x^2}u_0(x,y,t)\end{pmatrix}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 25e^{5x+5y-5t}\right]\right\} = 0. \tag{115}$$

$$N\left(\frac{\partial^2}{\partial y^2}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 5\begin{pmatrix}4u_0{}^3(x,y,t)\left(\frac{\partial}{\partial y}u_0(x,y,t)\right)^2\\ +u_0{}^4(x,y,t)\frac{\partial^2}{\partial y^2}u_0(x,y,t)\end{pmatrix}\right]\right\}$$
$$= L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 25e^{5x+5y-5t}\right]\right\} = 0. \tag{116}$$

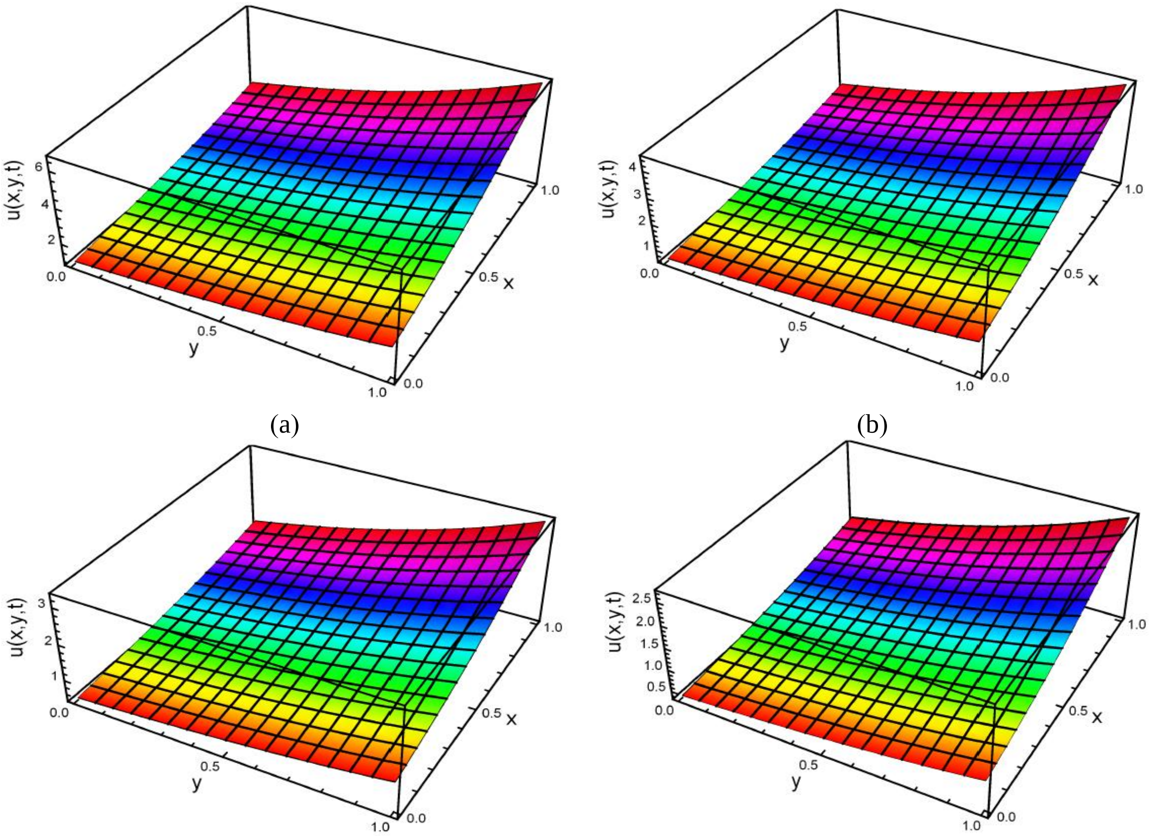
$$N\left(\frac{\partial^2}{\partial x\partial y}u_0(x,y,t)\right) = L_{xyt}{}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[25e^{5x+5y-5t} - 25e^{5x+5y-5t}\right]\right\} = 0. \tag{117}$$

$$N\left(\frac{\partial^2}{\partial x \partial t}u_0(x,y,t)\right) = N\left(\frac{\partial^2}{\partial y \partial t}u_0(x,y,t)\right)$$
$$= L_{xyt}^{-1}\left\{\frac{1}{(s^2-p^2-k^2)}L_{xyt}\left[-25e^{5x+5y-5t} + 25e^{5x+5y-5t}\right]\right\} = 0. \quad (118)$$

Therefore, $\left\|N\left(u_0''(x,y,t)\right)\right\| = \|0\| = 0 < \frac{1}{e}$.

In the same way, we get $\left\|N^{(3)}\left(u_0(x,y,t)\right)\right\| = \left\|N^{(4)}\left(u_0(x,y,t)\right)\right\| = \cdots = \left\|N^{(k)}\left(u_0(x,y,t)\right)\right\| = 0 < \frac{1}{e}$, for all $k \geq 0$.

The series solution $u(x,y,t) = \sum_{i=0}^{\infty} u_i\,(x,y,t)$ obtained by Daftardar-Gejji and Jafari (New iterative) approach is completely convergent on the domain of interest since the criteria of Theorem 16 are satisfied.



(a) (b)

(c) (d)

Figure 3 3D solution plots of Example 3 at a) t = 0.1, b) t = 0.5 c) t = 0.8, and d) t = 1.

Conclusions

In this work, the amalgamation of the 3-dimensional Laplace transform method and the new iterative (Daftardar-Gejji and Jafari) method is applied to find the exact solution of a 2-dimensional nonlinear Klein-Gordon equation with quadratic, cubic, and polynomial nonlinearity satisfying initial and boundary conditions. The basic definitions, properties, and theorems of the 3-dimensional Laplace transform and an Iterative method are discussed. Using the described method, the solution to the problem is obtained first, by applying the 3-dimensional Laplace transforms to the linear part of the equation and then implementing the iterative method to the non-linear part as introduced in section 3. Here the main goal of using Daftardar-Gejji and Jafari iterative method is to disappear the noise terms from the nonlinear part of the equation through a successive iteration process. Testing Examples are presented to demonstrate the liability and validity of the method under consideration. The obtained result by this proposed method strongly agrees with the exact solution obtained by other existing methods in the literature [7,11]. By this method, further non-trivial problems that were solved during earlier methods become trivial in the sense that the decomposition $u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots + u_m(x,y,t) + \cdots$ consists only of 1 term i.e, $u_n(x,y,t) = u_0(x,y,t)$. This technique is computationally efficient in these situations that are significant to many academics in the field of applied sciences. Moreover, the results from the iterative process converge to the exact solution. In conclusion, the 3-dimensional Laplace transform method coupled with the iterative method finds quite practical analytical results with less computational work, especially for the nonlinear partial differential equations. Due to its high importance, the author of this paper aimed as a future scope to apply this method to solve higher order applicable non-linear partial differential equations arising in various fields of sciences with different types of fractional operators.

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