A New Algorithm Based on Bernstein Polynomials Multiwavelets for the Solution of Differential Equations Governing AC Circuits

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Received: 6 June 2020, Revised: 6 May 2021, Accepted: 6 June 2021

Abstract

We extend the application of multiwavelet-based Bernstein polynomials for the numerical solution of differential equations governing AC circuits (LCR and LC). The operational matrix of integration is obtained from the orthonormal Bernstein polynomial wavelet bases, which diminishes differential equations into the system of linear algebraic equations for easy computation. It appeared that fewer wavelet bases gave better results. The convergence and exactness were examined by comparing the calculated numerical solution and the known analytical solution. The error function was calculated and illustrated graphically for the reliability and accuracy of the proposed method. The proposed method examined several physical issues that lead to differential equations.

Keywords: AC circuits, Bernstein polynomials multiwavelets, Operational matrix, Wavelet analysis

Introduction

A differential equation is a mathematical equation that depicts the connection between a function (physical values in the case of a physical model) and its derivatives. Differential equations can be used to solve issues in mathematical physics such as mechanics, elasticity, the linearized theory of water waves, stable steady-state heat conduction, and radioactive heat transfer. AC circuits include features that are used in a variety of everyday gadgets. Many numerical methods based on different operational matrices have been developed to determine solutions of differential equations with a given degree of accuracy [1-4] Wavelets were also used for solving several physical problems [5-8]. In this paper, we show the application of the proposed method in LCR (electric circuit comprising of resistance (R), capacitor (C) and inductor (L)) and L-C circuits (comprising of inductor (L) and capacitor (C)) circuits see **Figure 1**. We solved examples based on these circuits.



Figure 1 LCR and LC circuit diagram.

The resonance effect of the LC circuit has significant applications in communications systems and signal processing, such as the tuning of radio to a specific radio channel. LC circuits are set at resonance for that specific carrier frequency and used to pick out a signal from that particular radio station at a particular frequency from a more complex signal. Generating signals at that particular frequency is the key component in many applications filters, mixers, oscillators and tuners. The series and parallel resonant circuits give voltage and current magnification, respectively. When subject to a sinusoidal applied voltage $V(t) = V_0 \sin \omega t$, the differential equation governing the current flow in a series LCR circuit is;

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t.$$
(1)

The differential equation governing the current flow in a series LC circuit when subject to a sinusoidal applied voltage $V(t) = V_0 \sin \omega t$ is

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i = \omega V_0 \cos \omega t.$$
⁽²⁾

In this proposed algorithm, a wavelet-based numerical method on Bernstein polynomials multiwavelets to solve the differential equation is given. In this paper, firstly we integrate the given differential equation then remove the integral operators by approximating the given function with the help of the newly constructed operational matrix of integration based on Bernstein polynomials multiwavelets, which converts the above differential equation into a system of algebraic equations for easier computations. The first example describes how the proposed methodology fits for finding a solution of the LCR circuit and the second example describes the LC circuit and 1 general example is given to understand the proposed method better.

Wavelets and Bernstein polynomials multiwavelets

Wavelets have achieved the present growth as a result of various numerical investigations [9-13]. Wavelets are used to compress given information in a manner so that the resulting signal is a better representation of the information than the given signal was in the original form. The continuous variation of the translation and dilation parameters u and v gives the following continuous wavelet form [1]

$$\psi_{u,v}(t) = \left| u \right|^{-1/2} \psi\left(\frac{t-v}{u}\right), \quad u, v \in \mathbb{R}, \quad u \neq 0,$$
(3)

when these 2 parameters u and v are regulated to $u = 2^{-d}$, $v = n 2^{-d}$ then new discrete wavelet family is obtained from the above equation as $\psi_{d,i}(t) = 2^{d/2} \psi(2^d t - i)$; $d, i \in \mathbb{Z}$, and $\int_{\mathbb{R}} \psi(t) dt = 0$. Bernstein polynomials of order n characterized over the interval [0, 1] are given as:

$$B_{m,n}(t) = \binom{n}{m} t^m (1-t)^{n-m} , \ \forall \ m = 0, 1, 2, ..., n.$$
(4)

Using the Gram-Schmidt process n+1 orthonormal Bernstein polynomials for order n can be obtained easily from Bernstein polynomials $B_{m,n}(t)$ given in Eq. (4).

Eight orthonormal Bernstein polynomials of order 7, which are obtained from Bernstein polynomials $B_{m,n}(p)$ given in Eq. (4) by utilizing the Gram-Schmidt process are given below:

$$b_{0}(t) = \sqrt{15} (1-t)^{7}$$

$$b_{1}(t) = \sqrt{13} (t-1)^{6} (-1+15t)$$

$$b_{2}(t) = -\sqrt{11} (t-1)^{5} (1-28t+105 t^{2})$$

$$b_{3}(t) = 3(t-1)^{4} (-1+39t-273t^{2}+455t^{3})$$

$$b_{4}(p) = -\sqrt{7} (t-1)^{3} (1-48t+468t^{2}-1456t^{3}+1365t^{4})$$

$$b_{5}(p) = \sqrt{5} (t-1)^{2} (-1+55t-660t^{2}+2860t^{3}-5005t^{4}+3003t^{5})$$

$$b_{6}(p) = \sqrt{3} (1-61t+885t^{2}-5225t^{3}+15125t^{4}-22737t^{5}+17017t^{6}-5005t^{7})$$

$$b_{7}(p) = -1+63t-945t^{2}-5775t^{3}-17325t^{4}+27027t^{5}-21021t^{6}+6435t^{7}.$$

Bernstein polynomials orthonormal multiwavelets $\psi_{i,j}(t) = \psi(d, i, j, t)$ where d is dilation parameter assumes any positive integer, j = 0, 1, 2..., N is Bernstein polynomial degree, translation parameter $i = 0, 1, 2, ..., 2^d - 1$, and normalized time t. They are characterized on the interval [0, 1) as [14];

$$\psi_{i,j}(t) = \begin{cases} 2^{d/2} b_j (2^d t - i) & \frac{i}{2^d} \le t < \frac{i+1}{2^d} \\ 0 & \text{otherwise,} \end{cases}$$
(5)

where $b_j(t)$ represents an orthonormal Bernstein polynomial of order j. On taking j = 7 and dilation parameter d = 0 with the help of these 8 orthonormal Bernstein polynomials, we can calculate 8 Bernstein polynomials orthonormal multiwavelets,

$$\psi_{0,0}(t) = \begin{cases} b_0(t), & 0 \le t < 1\\ 0, & \text{otherwise} \end{cases}, \\ \psi_{0,1}(t) = \begin{cases} b_1(t), & 0 \le t < 1\\ 0, & \text{otherwise} \end{cases}, \\ \cdots \\ \psi_{0,7}(t) = \begin{cases} b_7(t), & 0 \le t < 1\\ 0, & \text{otherwise} \end{cases},$$

which are shown in Figure 2. Similarly, we can get $\psi_{i,j}(t)$ for different order j and dilation paramete d



Figure 2 Eight Bernstein polynomials orthonormal multiwavelets for j = 7, d = 0.

Function approximation

As $f(t) \in L^2[0,1]$, we may expand f(t) as:

$$f(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e_{ij} \psi_{ij}(t) , \qquad (6)$$

where $e_{ij} = \langle f(t), \psi_{ij}(t) \rangle$, here $\langle ., . \rangle$ denote the inner product on the Hilbert space $L^2(R)$. In Eq. (6) the infinite series is truncated at the levels $i = 2^d - 1$ and j = N, we obtain an approximate representation for f(t) as

$$f(t) \approx \sum_{i=1}^{2^{d}-1} \sum_{j=0}^{N} e_{ij} \psi_{ij}(t) = E^{T} \Psi(t),$$
(7)

where E and Ψ are $2^d (N+1) \times 1$ order matrices given by;

$$E = [e_{00}, \dots, e_{0N}; e_{10}, \dots, e_{1N}; \dots; e_{(2^d - 1)0}, \dots, e_{(2^d - 1)N}]^T,$$

$$\Psi(t) = \left[\psi_{00}(t), \dots, \psi_{0N}(t); \psi_{10}(t), \dots, \psi_{1N}(t); \dots; \psi_{(2^d - 1)0}(t), \dots, \psi_{(2^d - 1)N}(t)\right]^T$$

Bernstein polynomials multiwavelets operational matrix of integration

In this section, the operational matrix of integration P is derived. First, we find the matrix P then on transforming the differential equation into integral equations by integration, after that truncating orthogonal series and approximating various signals involved in the equation by using the operational matrix of integration to remove the integral operations. The elements are the basis functions, orthogonal on a certain interval [a,b]. The Haar wavelets-based operational matrix of integration was derived by [1]. The integration of $\Psi(t)$ is approximated by Bernstein polynomials orthonormal multiwavelets series by Bernstein polynomials orthonormal multiwavelets coefficient matrix P:

$$\int_{0}^{t} \Psi(t) dt = P_{2^{d}(N+1) \times 2^{d}(N+1)} \Psi(t),$$
(8)

where P is $2^{d}(N+1)$ -order Bernstein polynomials orthonormal multiwavelets based operational matrix of integration.

Method of solution

In this section, we find the solution of the differential equation,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t),$$

governing the flow of current in a series LCR circuit is given subject to an applied voltage $V(t) = V_0 \sin \omega t$ with boundary conditions q(t) = 0 and q'(t) = 0. On integrating Eq. (1) with respect to t and taking $V(t) = D^T \psi(t)$ we get;

$$L_{0}^{'}\frac{d^{2}q}{dt^{2}}dt + R_{0}^{'}\frac{dq}{dt}dt + \int_{0}^{t}\frac{q}{C}dt = \int_{0}^{t}D^{T}\psi(t)dt,$$
(9)

on taking

$$q(t) = E^{T} \psi(t)$$
(10)

we get

$$L\int_{0}^{t} \frac{d^{2}q}{dt^{2}} dt + R\int_{0}^{t} \frac{dq}{dt} dt + \frac{1}{C}\int_{0}^{t} q dt = \int_{0}^{t} D^{T} \psi(t) dt,$$
(11)

on using Eqs. (8) and (10) with boundary conditions Eq. (11) becomes;

$$L\frac{dq}{dt} + RE^{T}\psi(t) + \frac{1}{C}E^{T}P\psi(t) = D^{T}P\psi(t),$$
(12)

on integrating Eq. (12) with respect to $t_{;}$

$$L_{0}^{t} \frac{dq}{dt} dt + RE^{T} \int_{0}^{t} \psi(t) dt + \frac{1}{C} E^{T} P \int_{0}^{t} \psi(t) dt = D^{T} P \int_{0}^{t} \psi(t) dt,$$
(13)

again, using Eqs. (8) and (10) with boundary conditions;

$$L E^{T} \psi(t) + R E^{T} P \psi(t) + \frac{1}{C} E^{T} P^{2} \psi(t) = D^{T} P^{2} \psi(t), \qquad (14)$$

$$LE^{T} + RE^{T}P + \frac{1}{C}E^{T}P^{2} = D^{T}P^{2},$$
(15)

$$XE = Z,$$
 (16)

where
$$X = LI + R(P)^{T} + \frac{1}{C}(P^{2})^{T}; Z = (P^{2})^{T}D$$
,

and I stands for the identity matrix. Eq. (16) is a set of algebraic equations which can be solved for E, where P is $2^{d}(N+1)$ order square matrix, called Bernstein multiwavelets-based operational matrix. On putting the values of E into Eq. (10), we get the desired solution. Similarly, in the LC circuit case, we can obtain current by integrating Eq. (2) with respect to t and using initial conditions (see **Appendix**).

Illustrative examples

The subsequent examples are illustrated to show the effectiveness and steadiness of our algorithm. Note that in the first example, the series in Eq. (7) is truncated at the level j = 5 and j = 7, respectively in example 1, 2 and example 3. The exactness of the proposed method is shown by manipulating the absolute error. $\Delta \xi(t_k)$, which is given by:

$$\mathbf{E}(t) = \left| \xi(t_k) - \tilde{\xi}(t_k) \right|,$$
(17)

where $\tilde{\xi}(t_k)$ is the approximate solution calculated at the point t_k and the exact solution at the corresponding point is $\xi(t_k)$.

Example 1

Consider the differential equation governing the LCR circuit given in Eq. (1) with L = 1H, C = 1F, $R = 1\Omega$ and the source voltage $V(t) = V_0 \sin(\omega t)$ is:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0\sin(\omega t),$$

where $V_0 = 1$ V, $\omega = 2$ with boundary conditions q(t) = 0 and q'(t) = 0, whose exact solution is given by

$$q(t) = \frac{1}{13} \left[e^{-t/2} \left(2\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{14\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right) \right) - 2\cos 2t - 3\sin 2t \right].$$

Now, on using Bernstein polynomial multiwavelets operational matrix of integration we solve the above LCR circuit problem using Eqs. (9) to (16) and calculated the approximated solution q1(t) and q2(t) for J = 5; d = 0. Figures 3 and 4 are representing the exact and approximate solution and absolute error E1(t) = |q(t) - q1(t)|.



Figure 3 Comparison of solutions; exact solution q(t); approximated solution q l(t).



Figure 4 Absolute error E1(t) for J = 5; d = 0.

Table 1 Approximate and exact solution of examp	ole	1.
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р	Exact solutions $q(t)$	Approximate solution q l(t)	Absolute error E1(t) = q(t) - q1(t)
0.0	3.243478×10 ⁻⁴	3.243457×10^{-4}	2.092873×10 ⁻⁹
0.2	2.512962×10^{-3}	2.512991×10^{-3}	2.844036×10^{-8}
0.4	0.018583	0.018583	5.415651×10 ⁻⁹
0.6	0.056811	0.056811	5.226375×10^{-9}
0.8	0.119373	0.119373	2.857198×10 ⁻⁸
0.99	0.201866	0.201866	1.323026×10^{-7}

Example 2

Consider the LC circuit given in Eq. (2) with L = 5 H, $C = .2F, \omega = 5$, the source voltage is $V(t) = V_0 \cos(\omega t)$, where $V_0 = 1 \text{ V}$ with boundary conditions i(t) = 0 and i'(t) = 0 and the exact solution is given by $i(t) = (5\sin(t) - \sin(5t))/24$.

Now using Bernsteins polynomial multiwavelets operational matrix of integration, we solve this LC circuit problem and calculate approximated solutions il(t) and i2(t) for d = 0 and d = 1, respectively by utilizing the above algorithm from Eqs. (9) to (16). Figure 5 represents the exact and approximate solution. Figure 6 accounts for absolute errors; El(t) = |i(t) - il(t)| and E2(t) = |i(t) - i2(t)| for dilation parameter d = 0 and d = 1, respectively.



Figure 5 Comparison of solutions; exact solution i(t); approximated solution i1(t) (with dilation parameter d = 0) and i2(t) (with dilation parameter d = 1).



Figure 6 Comparison of absolute errors; E1(t) and E2(t) for dilation parameter d = 0 and d = 1

р	Exact solutions <i>i</i> (<i>p</i>)	Approximate solution <i>i</i> 1(<i>p</i>) (for <i>d</i> = 0)	Approximate solution <i>i</i> 2(<i>p</i>) (for <i>d</i> =1)
0.0	0.0	0.000000	-0.000000
0.2	0.006328	0.006263	0.006330
0.4	0.043241	0.043260	0.043240
0.6	0.111754	0.111669	0.111758
0.8	0.180983	0.180955	0.180975
0.99	0.215262	0.216178	0.215250

Table 2 Approximate and exact solution of example 2.

Example 3

Consider the problem,

$$\frac{1}{2}y'(p) + y(p) = u(p)$$
(18)

with y(0) = 0

where u(p) is p^4 . The analytic solution of Eq. (18) is

$$y(p) = \frac{3}{2} (1 - \exp(-4p)) + p^{3} (p-2) + 3p (p-1).$$

We solve this by Bernstein polynomials multiwavelets, with j = 7; d = 0. We assume the unknown function y(p) is given by $y(p) = C^T \psi(p)$, to get the solution of the differential equation; take

$$y(p) = C^T \psi(p), u(p) = D^T \psi(p)$$
.

On integrating Eq. (18), we get

$$\frac{1}{2}C^{T}\psi(p) = -\int_{0}^{p}C^{T}\psi(p)dp + \int_{0}^{p}D^{T}\psi(p)dp,$$

$$\frac{1}{2}C^{T} + C^{T}P = D^{T}P,$$

$$LC = M,$$
(19)

where $L = \frac{1}{2}I + P^{T}$; $M = P^{T}D$ and I is the identity matrix Eq. (19) is a set of algebraic equations which gives the matrix C where P is $2^{d}(N+1)$ order square matrix, called Bernstein multiwaveletsbased operational matrix. After getting the value of C, we get the desired solution. **Figure** 7 and **8** represents the approximate solution and absolute errors, respectively.

p	Exact solution $\xi l(p)$	Approximate solutions ξ2(p)	Approximate error $E(p) = \left \xi l(p) - \xi 2(p)\right $
0.0	0.0	2.814345×10^{-7}	2.814345×10^{-7}
0.2	1.199309×10^{-4}	1.199842×10^{-4}	5.328864×10^{-8}
0.4	3.606554×10^{-3}	3.606552×10^{-3}	2.230206×10^{-9}
0.6	0.025809	0.025809	1.983451×10 ⁻⁸
0.8	0.102755	0.102755	6.534773×10^{-8}
0.99	0.283194	0.283194	1.156281×10^{-7}

Table 3 Approximate and exact solution of example 3.



Figure 7 Comparison of solutions; exact solution $\xi l(p)$; approximated solution $\xi 2(p)$ (for d = 0).



Figure 8 Absolute error E(t) for j = 7, d = 0.

Conclusions

We have used Bernstein polynomials multiwavelets to construct the operational matrix of integration with 2 dilation parameters d = 0 and d = 1. Furthermore, our method illustrates the comparison of 2 distinct dilation parameters' solutions. The method is simple and clear due to the use of only 6 orthonormal polynomials of degree 5.

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Appendix: Algorithm for the numerical method

Step1: Obtain the orthonormalized version $b_n(u)$ of Bernstein polynomials $B_{i,n}(u)$.

1) The Bernstein polynomial of degree N is $B_{j,N}(y)$

$$b_0(y) := \frac{B_{0,N}(y)}{\|B_{0,N}(y)\|}.$$

2) For j=1 to N

3)

$$C_{j,N}(y) \coloneqq B_{j,N}(y) - \frac{\langle B_{j,N}(y), C_{j-1,N}(y) \rangle}{\|C_{j-1,N}(y)\|} * C_{j-1,N}(y) - \frac{\langle B_{j,N}(y), C_{j-2,N}(y) \rangle}{\|C_{j-2,N}(y)\|} * C_{j-2,N}(y)$$
$$\dots - \frac{\langle B_{j,N}(y), B_{0,N}(y) \rangle}{\|B_{0,N}(y)\|} * B_{0,N}(y).$$
$$b_{j}(y) \coloneqq \frac{C_{j,N}(y)}{\|C_{j,N}(y)\|}$$

is the required Bernstein polynomial. Here $\langle p(y), q(y) \rangle$ represents the inner product of p(y) and q(y) in the Hilbert space $L^2(R)$ and $\|\xi(y)\|$ is the L^2 norm of the given function $\xi(y)$.

Step2: Evaluate $\psi_{i,j}(t)$ over the interval [0, 1) using $b_j(t)$ obtained from step 1.

Step3: Manipulate operational matrices P and matrix D^{T} .

Step4: On integrating given differential equation governing AC circuit with respect to t and taking Voltage function $V(t) = D^T \psi(t)$, $q(t) = E^T \psi(t)$ and $\int_{0}^{t} q dt = E^T P \psi(t)$ it converts to an algebraic system which can be easily solved for E.

Step 5: Put values of E^{T} from step 4 to get the desired solution function is $q(t) = E^{T} \psi(t)$.