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Drag Over A Fluid Sphere Filled With Couple Stress Due to Flow of A Couple Stress Fluid with Slip Condition

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Abstract

In this article, the exact solution for a couple stress fluid flow past a fluid sphere filled with a couple stress fluid is considered using interfacial slip on the boundary. The velocity is expressed with regard to the stream function. The external velocity and internal velocity are alongside the drag coefficient. It was noticed that with the increases in slip parameter, the drag coefficient is decreasing and the inflow pattern has more circulations with more area appearing near the poles and internal flow disappears gradually. The special cases of viscous fluid, no-slip condition, and solid sphere are evaluated. A good understanding of the current and literature outcomes have been obtained, including the special case of a viscous fluid past a couple stress fluid sphere.

Keywords: Couple stress fluid, Slip condition, Gegenbauer polynomials, Non-newtonian fluid, Creeping flow, Drag, Stream function

Introduction

Couple stress fluids are widely used for biofluids such as blood, moisturizers with tiny amounts of polymer additives, synthetic fluids, and electro-rheological fluids. The important characteristic of stress-coupled liquids is that their stress tensor is anti-symmetric when classical Newtonian theory is applied it cannot predict their exact flow characteristics. Stokes [1] obtained the expression for the distribution of velocity and pressure in the work of stress fluid coupling through a sphere. It is observed that the effect of the coupling stress is equivalent to a significant increase in viscosity. Ramkissoon [2] proposed a general formula for the resistance of an axisymmetric body, which restricts the fluid flow function at a couple of stress fluids. Iyengar and Srinivasacharya's [3] study on the couple stress fluid flowing through an approximate sphere, represents the flow model of the exterior and interior zones respectively. The influence of permeability parameters and geometric parameters on the resistance is analyzed numerically. Aparna and Ramana Murthy's [4] study on the uniform flow of a fluid constrained by incompressible torque through a permeable sphere has found a solution by using conditions suitable for the movement of strained fluid coupled under different flow conditions. The discussion found the study by Umavathi *et al.* [5] is very interesting as it focuses on the new method that analyzes strong and weak flows with comparable porosity conditions involved with couple stress fluid parameters. Iyengar and Geetha Vani [6] have used the Stokes method to study the slow rotation of a region filled with couple stress fluids between 2 confocal oblate spheroids. They observed that there is a direct relation between rotation parameter and couple on the spheroid. Waqar khan and Faisal [7] obtained an analytic solution for the velocity components: A couple stress fluid flow over a 2D grid; a reverse flow over a plate; a stretching surface; and flow over a corner. Ashmawy's [8] study on the couple stress fluid passing through a rigid sphere with a linear slip condition reflects that while the couple stress condition vanished on the sphere surface, the drag coefficient was thus evaluated. Rahmat Ellahi *et al.* [9] examined the effect of peristaltic transmission of nanofluids on chemical reactions and energy activation. Recently, Krishnan and Pankaj Shukla [10] have obtained an analytic solution for laminar flow of couple stress fluid flow past a fluid sphere inside a solid core with the no-slip condition on the boundary. In the effect of couple stress, they observed that drag, velocity, and pressure to be decreasing. Very recently, Alsudais *et al.* [11] have studied the flow of couple stress between 2 eccentric spherical boundaries with the no-slip condition. They obtained results using the semi-analytic method and observed that with an increase in couple stress parameters there is an increase in drag force. All the above investigations are about couple stress fluid studies over different geometries with the no-slip condition. Basset [12] calculated the resistance on the sphere as a function of the slip parameter $s=\frac{\beta a}{\mu}$. Happel and

Brenner [13] in their monograph reviewed the creeping flow of viscous fluid over a fluid sphere with interfacial slip condition on the surface. The special cases of slip parameter condition ($s\to\infty$) lead to a no-slip case, the viscosity ratio given as $\mu=\frac{\mu_i}{\mu_e}$, where $\mu_{i,}$ μ_e stands for internal and external viscosities. Further when $\mu\to\infty$ fluid sphere tends to be a solid sphere. Devakar *et al.* [14] obtained an analytic solution for the flow of couple stress fluid between parallel plates using slip condition for Poiseuille, Couette, and generalized Couette flows. They observed that velocity is less affected by a rise in slip on the boundary. Feng and Michaelides [15] have found the directing equation of the dimensionless problem with slip condition and solved it using the formula of the flow-vortex function. The equation can be solved using the stretched coordinate system and the tri-diagonal matrix algorithm. Murthy and Meduri [16] considered the viscous flow through the contaminated fluid sphere with interface slip condition and calculated the shear stress and resistance on the surface. The special cases, such as the no-slip case and solid sphere case were also deduced. The results obtained in special cases were matched with results reported in Happel ad Brenner [13]. These authors have considered slip condition and studied the viscous flow over a solid sphere, viscous drop, partially contaminated viscous drop, etc., The main objective of the current study is obtained after reviewing the literature on interfacial slip condition over a fluid sphere with internal and external couple stress fluid that has not yet been reported so far. In this manuscript, we have considered the flow of a couple stress fluid over a stationary fluid sphere/drop with a couple stress fluid inside it. Interfacial slip, regularity condition, and shear stress continuity are considered as the boundary. The analytic solution has been obtained for the drag over the surface using the above-mentioned conditions. In addition, the case of viscous flow past a couple stress fluid sphere is also studied. The manuscript is organized as follows: Definition of the problem, solution, and calculation of drag force. The exact solutions for viscous fluid flow past a couple stress fluid sphere, results, and discussion, followed by the conclusion.

Materials and methods

Definition of problem

Consider a fluid sphere filled with couple stress fluid which is placed fixed in a uniform flow of couple stress fluid. The flow is assumed to be steady, incompressible, and axisymmetric. The geometry of the model is presented in **Figure 1**.

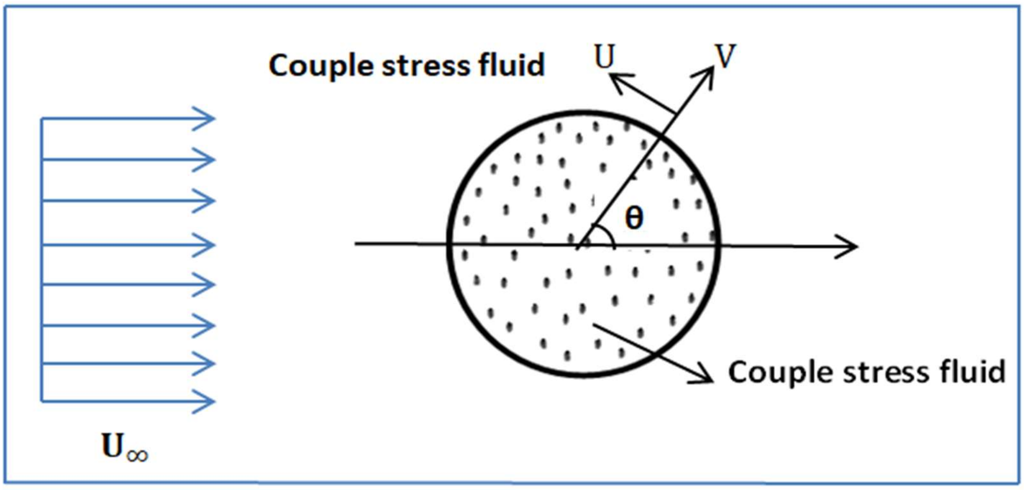


Figure 1 Flow geometry of couple stress fluid over a couple stress fluid sphere.

The continuity equation is;

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{q}) = 0. \tag{1}$$

For incompressible flow, Eq. (1) reduces to;

$$\nabla.\bar{q} = 0. \tag{2}$$

The field equations governing the couple stress fluid flow are given by;

$$\rho\frac{d\bar{q}}{dt} = \rho\bar{f} - \nabla p + \mu(\nabla \times \nabla \times \bar{q}) - \eta(\nabla \times \nabla \times \nabla \times \nabla \times \overline{q}) \tag{3}$$

where p is hydro-static pressure at any point, ρ is the density of the fluid, $\bar{q}$ is the velocity of the fluid, μ is viscosity co-efficient, $\bar{f}$ is body force per unit mass, η is couple stress viscosity coefficient.
For steady flow and in absence of body forces, the Eq. (3) reduces to;

$$\nabla p = \mu(\nabla \times \nabla \times \bar{q}) - \eta(\nabla \times \nabla \times \nabla \times \nabla \times \overline{q}). \tag{4}$$

Due to the geometrical shape of the present problem, we choose the spherical coordinate system for reference. The scale factors for the system are $h_1 = 1,\ h_2 = R,\ h_3 = R\ sin\theta$.

Since the flow is axisymmetric, the velocity vector is chosen in the form;

$$\bar{q} = U(r,\theta)\bar{e}_r + V(r,\theta)\bar{e}_\theta. \tag{5}$$

The following non-dimensional scheme is taken;

R= a r; $\Psi = \psi U_\infty a^2$; P = $p\frac{U_\infty \mu}{a}$; $E_0^2 = \frac{E^2}{a^2}$; U= $u U_\infty$; V=vV_∞.

Now, the velocity will be represented in the form of stream function ψ as;

$$U(r,\theta) = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}\ ;\ V(r,\theta) = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{6}$$

Thus, Eq. (4) reduce to;

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2\sin\theta}\frac{\partial(E^2\psi)}{\partial\theta} - \frac{\eta}{r^2\sin\theta}\frac{\partial(E^4\psi)}{\partial\theta} \tag{7}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{\mu}{r\sin\theta}\frac{\partial(E^2\psi)}{\partial r} - \frac{\eta}{r\sin\theta}\frac{\partial(E^4\psi)}{\partial r} \tag{8}$$

where,

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}. \tag{9}$$

Eliminating the pressure, an equation for stream function (Aparna and Murthy [4]) is of the form;

$$E^4[E^2 - \lambda^2]\,\psi(r,\theta) = 0;\ \text{where } \lambda^2 = \frac{\mu a^2}{\eta}. \tag{10}$$

The couple stress fluid reduces to viscous fluid, when the geometric parameter $\lambda \to \infty$.

The solutions of ψ which are regular for external flow(ψ_e) and for internal flow(ψ_i) regions are given by;

$$\psi_{\mathrm{e}} = \left[\mathrm{r}^2 + \frac{\mathrm{A}_1}{\mathrm{r}} + \mathrm{B}_1\mathrm{r} + \mathrm{C}_1\sqrt{\mathrm{r}}\ \mathrm{K}_{\frac{3}{2}}(\lambda_{\mathrm{e}}\mathrm{r})\right]\mathrm{G}_2(x) \ \text{ and} \tag{11}$$

$$\psi_{\mathrm{i}} = \left[\mathrm{A}_2\,\mathrm{r}^2 + \mathrm{B}_2\mathrm{r}^4 + \mathrm{C}_2\sqrt{\mathrm{r}}\,\mathrm{I}_{\frac{3}{2}}(\lambda_i\mathrm{r})\right]\ \mathrm{G}_2(\mathrm{x}) \tag{12}$$

where $\mathrm{K}_{\frac{3}{2}}(\lambda_{\mathrm{e}}\mathrm{r})$ and $\mathrm{I}_{\frac{3}{2}}(\lambda_i\mathrm{r})$ are modified Bessel's functions of order $\frac{3}{2}$ and $G_2(\mathrm{x}) = \frac{1}{2}\,(1-\mathrm{x}^2)$ is Gegenbauer polynomial of order 2 (Aparna and Murthy [4], Noura S. Alsudais *et al.* [11]).

The parameters A_1, B_1, C_1, A_2, B_2, C_2 are obtained by implementing the following boundary conditions on Eqs. (11) - (12).

(i). Regularity conditions: $\lim_{\mathrm{r}\to\infty}\psi_{\mathrm{e}} = \frac{1}{2}\mathrm{r}^2\sin^2\theta$ (outside the region) and;

$$\lim_{\mathrm{r}\to 0}\psi_{\mathrm{i}} = \text{Finite (inside the region).} \tag{13}$$

(ii). Impermeability condition i.e., no mass transfer at the interface of fluid sphere;

$$\psi_{\mathrm{e}} = \psi_{\mathrm{i}} = 0. \tag{14}$$

(iii). Slip condition: The tangential velocity of the liquid relative to the solid at a point on the surface is relative to the tangential stress acting at that point (Happel and Brenner [13]) i.e,;

$$\tau_{\mathrm{r}\theta} = \beta(q_\theta - V_\theta). \tag{15}$$

(iv). Shear stress continuous at the interface of the viscous sphere i.e.;

$$\tau_{\mathrm{r}\theta\mathrm{e}} = \tau_{\mathrm{r}\theta\mathrm{i}}. \tag{16}$$

(v). Type A condition: Vanishing of couple stress on the boundary;

$$\frac{\partial[E^2\psi]}{\partial r} = \left(\frac{1}{r} + e\right)E^2\psi \ \text{ where } \mathrm{e} = \frac{\eta'}{\eta}. \tag{17}$$

Solution of the problem

Using the boundary conditions (13) - (17) in Eqs. (11) - (12), the following system of equations are obtained;

$$\mathrm{A}_1 + \mathrm{B}_1 + C_1{}' = -1,\ \mathrm{A}_2 + \mathrm{B}_2 + C_2{}' = 0,$$

$$(6+\mathrm{s})\mathrm{A}_1 - \mathrm{s}\mathrm{B}_1 + \mathrm{C}_1'\{4 + (2+\mathrm{s})\Delta_1(\lambda_{\mathrm{e}})\} + 2\mathrm{s}\mathrm{A}_2 + 4\mathrm{s}\,\mathrm{B}_2 - \mathrm{C}_2'\{\mathrm{s}\,\Delta_2(\lambda_{\mathrm{i}})\} = 2\mathrm{s},$$

$$-6\mathrm{A}_1 - \mathrm{C}_1'\{4 + 2\,\Delta_1(\lambda_{\mathrm{e}})\} + 6\,\mu\,\mathrm{B}_2 + \mu\,\mathrm{C}_2'\{4 + 2\Delta_2(\lambda_{\mathrm{i}})\} = 0,$$

$$(4+2\mathrm{e})\mathrm{B}_1 - \lambda_e^2\mathrm{C}_1'\{\Delta_1(\lambda_{\mathrm{e}}) + (1+\mathrm{e})\} = 0,$$

$$(10-10\mathrm{e})\mathrm{B}_2 - \lambda_i^2\mathrm{C}_2'\{\Delta_2(\lambda_{\mathrm{i}}) + (1+\mathrm{e})\} = 0,$$

where, $\mathrm{C}_1' = \mathrm{C}_1\,\mathrm{K}_{\frac{3}{2}}(\lambda_e)$, $\mathrm{C}_2' = \mathrm{C}_2\,\mathrm{I}_{\frac{3}{2}}(\lambda_i)$, slip parameter $s = \frac{\beta a}{\mu}$, viscosity ratio $\mu = \frac{\mu_i}{\mu_e}$, and couple stress parameter $e = \frac{\eta'}{\eta}$.

Solving the above equations analytically, resulted;

$$\mathrm{A}_1 = -1 - (1+\chi_1)\mathrm{C}_1',\ \mathrm{B}_1 = \chi_1\mathrm{C}_1' \quad \text{and} \quad \mathrm{C}_1' = \frac{(3\mathrm{s}+6)\mathrm{g}_2 + 3\mathrm{k}_2}{\pi}.$$

$A_2 = -(1+\chi_2)C_2'$, $B_2 = \chi_2 C_2'$ and $C_2' = \frac{-(3s+6)g_1 - 3k_1}{\pi}$.

where, $\pi = k_1 g_2 - k_2 g_1$

$k_1 = (-2-6\chi_1) - (1+2\chi_1)s + (2+s)\Delta_1(\lambda_e)$;

$k_2 = 2s(-1+\chi_2) - s\,\Delta_2(\lambda_i)$;

$g_1 = (1+3\chi_1) - \Delta_1(\lambda_e)$;

$g_2 = \mu(2+3\,\chi_2) + \mu\,\Delta_2(\lambda_i)$;

$\chi_1 = \frac{\lambda_e^2[\Delta_1(\lambda_e)+(1+e)]}{4+2e}$; $\chi_2 = \frac{\lambda_i^2[\Delta_2(\lambda_i)+(1+e)]}{10-10}$.

Thus external and internal flow stream functions are derived.

Special cases

For external stream function;

Couple stress fluid tends to viscous fluid, in the limiting case, when $\lambda_i \to \infty, \lambda_e \to \infty \implies \chi_1 \to \infty$, $\chi_2 \to \infty$ and hence, we get;

$C_1 = 0$, $B_1 = -\frac{(3s+6)\mu+2s}{(2s+6)\mu+2s}$ and $A_1 = \frac{\mu s}{(2s+6)\mu+2s}$.

Therefore, the external stream function becomes;

$$\psi_e = \left(r^2 + \left(\frac{\mu s}{(2s+6)\mu+2s}\right)\left(\frac{1}{r}\right) - \left(\frac{(3s+6)\mu+2s}{(2s+6)\mu+2s}\right) r\right) G_2(x). \tag{18}$$

When $s \longrightarrow \infty$, we obtained, the no-slip condition, and hence, we get external velocity for a fluid sphere with no-slip condition as;

$$\psi_e = \left(r^2 + \left(\frac{\mu}{2\mu+2}\right)\left(\frac{1}{r}\right) - \left(\frac{3\mu+2}{2\mu+2}\right) r\right) G_2(x). \tag{19}$$

Also, when $\mu \longrightarrow \infty$, viscous fluid sphere reduces to a solid sphere and hence, we get the external velocity for a solid sphere with no-slip condition as;

$$\psi_e = \left(r^2 + \left(\frac{1}{2}\right)\left(\frac{1}{r}\right) - \left(\frac{3}{2}\right) r\right) G_2(x). \tag{20}$$

This expression for ψ_e in eq. (20) comes to agree with the relation for solid sphere with no-slip on as derived by Happel and Brenner [13].

For internal stream function;

Couple stress fluid tends to viscous fluid in the limiting case, when $\lambda_i \to \infty, \lambda_e \to \infty$ which implies $\chi_1 \to \infty$, $\chi_2 \to \infty$ hence, we get;

$C_2 = 0$, $B_2 = \frac{s}{(2s+6)\mu+2s}$ and $A_2 = -\frac{s}{(2s+6)\mu+2s}$.

Accordingly, the internal stream function becomes as;

$$\psi_i = \left(\left(\frac{-s}{(2s+6)\mu+2s}\right) r^2 + \left(\frac{s}{(2s+6)\mu+2s}\right) r^4\right) G_2(x). \tag{21}$$

when $s \longrightarrow \infty$, we obtain, no-slip condition and hence, resulting in the internal stream function for a viscous flow past a fluid drop with a no-slip condition as;

$$\psi_i = \left(\left(-\frac{1}{2\mu+2}\right) r^2 + \left(\frac{1}{2\mu+2}\right) r^4\right) G_2(x). \tag{22}$$

Eqs. (18) and (21) which corresponds with of Murthy and Meduri [16].

Eqs. (19) and (22) approved with Happel and Brenner [13], Murthy and Meduri [17] results.

Again, when $\mu \longrightarrow \infty$, viscous fluid sphere reduces to a solid sphere, and hence, in this case, it reduces to zero which is the same as that of an internal stream function with no-slip for a solid sphere. This is because in this case there will be no internal flow.

Drag force on a sphere

The drag force on the fluid sphere is given in limit as;

$$\mathrm{F_z} = 8\pi\mu \lim_{r \longrightarrow \infty} \left[\frac{\mathrm{R}(\Psi_e^* - \Psi_\infty^*)}{\mathrm{R}^2 \sin^2 \theta}\right] \tag{23}$$

where Ψ_∞^* denotes the stream function correlate to the fluid motion at infinity, then the stream function $(\Psi_e^* - \Psi_\infty^*)$ gives a state of rest at infinity Happel and Brenner [13].

Applying the limit, we get;

$$\mathrm{F_z} = (4\pi\mu \mathrm{U_\infty}\, a)\mathrm{B_1}. \tag{24}$$

Case (i): We know, that as $\lambda_e \longrightarrow \infty, \lambda_i \longrightarrow \infty$ and $\chi_1 \longrightarrow \infty, \chi_2 \longrightarrow \infty$;

$$\mathrm{F_z} = -(4\pi\mu \mathrm{U_\infty}\, a)\left[\frac{(3s+6)\mu + 2s}{(6+2s)\mu+2s}\right].$$

Case (ii): If $s \longrightarrow \infty$, then fluid sphere with no-slip condition case is obtained;

$$\mathrm{F_z} = -(6\pi\mu \mathrm{U_\infty}\, a)\left[\frac{\mu+\frac{2}{3}}{\mu+1}\right], \quad \text{here } \mu = \frac{\mu_i}{\mu_e}.$$

This matches with result obtained by Happel and Brenner [13]. Eqs. (4) - (21) for fluid sphere with the case of no-slip condition.

Case (iii): when $\mu \longrightarrow \infty$ we get the case of solid sphere.

$$\mathrm{F_z} = -(4\pi\mu \mathrm{U_\infty}\, a)\left[\frac{3s+6}{2s+6}\right]$$

on simplifying we get;

$$\mathrm{F_z} = -(6\pi\mu \mathrm{U_\infty}\, a)\left[\frac{s+2}{s+3}\right]$$

which is the drag force for a viscous fluid flow past a solid sphere with slip condition, which is in agreement with the results of Happel and Brenner [13].

Now, when $s \longrightarrow \infty$, we get no-slip condition case and hence the drag force will take the form;

$$\mathrm{F_z} = -(6\pi\mu \mathrm{U_\infty}\, a).$$

This is the drag force for a viscous fluid flow past a fluid sphere with no-slip condition. This is also matching with the results of Ashmawy [8].

The drag coefficient is $\mathrm{Cd} = \frac{\mathrm{F_z}}{\frac{1}{2}\rho\, \mathrm{U_\infty^2} \pi\, a^2}$;

$$= \frac{-(6\pi\mu \mathrm{U_\infty}\, a)}{\frac{1}{2}\rho\, \mathrm{U_\infty^2}\pi\, a^2} = -\frac{24}{\mathrm{Re}} \quad \text{with } \mathrm{Re} = \frac{\rho\, \mathrm{U_\infty}(2a)}{\mu}.$$

This is the drag coefficient for a solid sphere with no-slip condition, which fits with Stokes case in Happel and Brenner [13].

Exact solutions for viscous fluid flow past a couple stress fluid sphere

The solutions ψ of eq. (10) which are regular for external viscous fluid flow (ψ_e') and internal couple stress fluid flow (ψ_{i}) are;

$$\psi_e' = \left[\mathrm{r}^2 + \frac{\mathrm{A}_1}{\mathrm{r}} + \mathrm{B}_1\mathrm{r}\right]\mathrm{G}_2(\mathrm{x}) \text{ and} \tag{25}$$

$$\psi_{\mathrm{i}} = \left[\mathrm{A}_2\,\mathrm{r}^2 + \mathrm{B}_2\mathrm{r}^4 + \mathrm{C}_2\sqrt{\mathrm{r}}\;\mathrm{I}_{\frac{3}{2}}(\lambda_{\mathrm{i}}\mathrm{r})\right]\mathrm{G}_2(\mathrm{x}) \tag{26}$$

where $\mathrm{I}_{\frac{3}{2}}(\lambda_{\mathrm{i}}\mathrm{r})$ is modified Bessel's function of order $\frac{3}{2}$ and $G_2(\mathrm{x}) = \frac{1}{2}(1 - \mathrm{x}^2)$ is Gegenbauer polynomial of order 2. [Noura S. Alsudais *et al.* [11]].

The parameters A_1, B_1, A_2, B_2, C_2 in Eqs. (25) - (26) are evaluated by using boundary conditions (13) - (17). Here we get the following system of equations as;

$$\mathrm{A}_1 + \mathrm{B}_1 = -1,$$

$$\mathrm{A}_2 + \mathrm{B}_2 + \mathrm{C}_2' = 0,$$

$$(4+\mathrm{s})\mathrm{A}_1 - (\mathrm{s}+2)\mathrm{B}_1 + 2\mathrm{s}\mathrm{A}_2 + 4\mathrm{s}\,\mathrm{B}_2 - \mathrm{s}C_2'\Delta_2(\lambda_{\mathrm{i}}) = 2 + 2\mathrm{s},$$

$$-6\mathrm{A}_1 + \mu\{6\mathrm{B}_2 + \mathrm{C}_2'[4 + 2\Delta_2(\lambda_{\mathrm{i}})]\} = 0,$$

$$(10 - 10\mathrm{e})\mathrm{B}_2 - \mathrm{C}_2'\lambda_i^2[\Delta_2(\lambda_{\mathrm{i}}) + (1+\mathrm{e})] = 0,$$

where $C_2' = \mathrm{C}_2\mathrm{I}_{\frac{3}{2}}(\lambda_i)$.

Solving the above equations, we get;

$$\mathrm{C}_2' = \left[\frac{\mathrm{s}}{\{[(2\mathrm{s}+6)\mathrm{z}_1) + \mathrm{s}(-2 + 2\chi_2 - \Delta_2(\lambda_{\mathrm{i}})]\}}\right]$$

$$\mathrm{B}_2 = \chi_2\mathrm{C}_2',$$

$$\mathrm{A}_2 = -(1 + \chi_2)\mathrm{C}_2',$$

$$\mathrm{B}_1 = -1 - \mathrm{z}_1\mathrm{C}_2',$$

$$\mathrm{A}_1 = -1 - \mathrm{B}_1 = \mathrm{z}_1\mathrm{C}_2',$$

where $\mathrm{z}_1 = \frac{\mu(3\chi_2 + 2 + \Delta_2(\lambda_{\mathrm{i}}))}{3}$, here $\chi_2 = \frac{\lambda_i^2[\Delta_2(\lambda_{\mathrm{i}}) + (1+\mathrm{e})]}{(10-10\mathrm{e})}$.

Thus the stream functions for external and internal flows are obtained.

Special cases

For external stream function;

when $\lambda_{\mathrm{i}} \to \infty$ we get,

$$\mathrm{B}_1 = -\frac{(3\mathrm{s}+6)\mu + 2\mathrm{s}}{(2\mathrm{s}+6)\mu + 2\mathrm{s}} \text{ and } \mathrm{A}_1 = \frac{\mu\mathrm{s}}{(2\mathrm{s}+6)\mu + 2\mathrm{s}}$$

Here by substituting A_1, B_1 in Eq. (25) the external stream function for viscous fluid past a viscous fluid with slip condition is obtained as;

$$\psi_e' = \left(r^2 + \left(\frac{\mu s}{(2s+6)\mu+2s}\right)\left(\frac{1}{r}\right) - \left(\frac{(3s+6)\mu+2s}{(2s+6)\mu+2}\right) r\right) G_2(x) \tag{27}$$

Also, when $s \longrightarrow \infty$, we get the no-slip condition, according external velocity for a fluid sphere with no-slip is being achieved.

$$\psi_e' = \left(r^2 + \left(\frac{\mu}{2\mu+2}\right)\left(\frac{1}{r}\right) - \left(\frac{3\mu+2}{2\mu+2}\right) r\right) G_2(x) \tag{28}$$

Also, when $\mu \longrightarrow \infty$, viscous fluid sphere reduces to a solid sphere and hence, we get external velocity for a solid sphere with no-slip condition;

$$\psi_e' = \left(r^2 + \left(\frac{1}{2}\right)\left(\frac{1}{r}\right) - \left(\frac{3}{2}\right) r\right) G_2(x) \tag{29}$$

For internal stream function;

We know, couple stress fluid tends to viscous fluid, when $\lambda_i \to \infty$, imples $z_1 \to \infty$, $\chi_2 \to \infty$, $\Longrightarrow$ $C_2 = 0$, $B_2 = \frac{s}{(2s+6)\mu+2s}$ and $A_2 = -\frac{s}{(2s+6)\mu+2s}$

Hence, internal stream function for viscous flow past a viscous drop with a slip condition will be;

$$\psi_i = \left(\left(-\frac{s}{(2s+6)\mu+2s}\right) r^2 + \left(\frac{s}{(2s+6)\mu+2s}\right) r^4\right) G_2(x) \tag{30}$$

Also, when $s \longrightarrow \infty$, the no-slip condition is achieved such as the internal velocity for a fluid sphere with no-slip condition will be;

$$\psi_i = \left(\left(-\frac{1}{2\mu+2}\right) r^2 + \left(\frac{1}{2\mu+2}\right) r^4\right) G_2(x) \tag{31}$$

Eqs. (27) and (30) which corresponds with Murthy and Meduri [17].

Eqs. (28) - (29) and (31) approved with Happel and Brenner [13] results.

Also, when $\mu \longrightarrow \infty$, viscous fluid sphere reduces to solid sphere such as the internal velocity for a solid sphere with no-slip condition becomes zero.

Drag force over a couple stress fluid drop;

The drag force on fluid sphere in the limiting form as given by Ashmawy [8] is;

$$F_z = 8\pi\mu \lim_{r \longrightarrow \infty} \left[\frac{R(\Psi_e^* - \Psi_\infty^*)}{R^2 \sin^2\theta}\right]. \tag{32}$$

Substituting above in Eq. (27) and simplifying we get;

$$F_z = (4\pi\mu U_\infty a) B_1.$$

Under special conditions $\lambda_i \to \infty$, $s \longrightarrow \infty$ and $\mu \longrightarrow \infty$, we get;

$$F_z = -(6\pi\mu U_\infty a)$$

which matches with the Eq. (4) of Ashmawy [8].

The drag coefficient, under special cases, reduces to;

$$Cd = -\frac{24}{Re}$$

Representing the drag coefficient with no-slip condition on solid sphere, which agree with Stokes case.

Results and discussion

The external and internal stream functions of Eq. (10) are computed using the boundary conditions (13) - (17). The drag coefficient is evaluated and its variations related to diverse slip parameter and couple stress parameter values are presented in the following graphs.

Figure 2: The variation of drag coefficient w.r.t. slip parameter for different viscosity coefficients is illustrated. **Table 1** gives the numerical values. It was observed that as the slip parameter raises the drag coefficient (Cd) decreases. Also, as the value of the ratio of couple stress viscosity coefficients (e) raises the drag coefficient values decreases.

Table 1 Drag coefficient (Cd) values for varying the slip parameter (s) at different couple stress viscosity coefficiets (e).

SI.No	$e\backslash s$	$s=10$	$s=20$	$s=30$	$s=40$	$s=50$	$s=60$	$s=70$
1	$e=2$	0.3115	0.2852	0.2764	0.2721	0.2694	0.2677	0.2665
2	$e=4$	0.3091	0.2822	0.2733	0.2689	0.2662	0.2645	0.2632
3	$e=6$	0.3080	0.2809	0.2719	0.2674	0.2647	0.2630	0.2617
4	$e=8$	0.3074	0.2801	0.2711	0.2666	0.2639	0.2621	0.2608

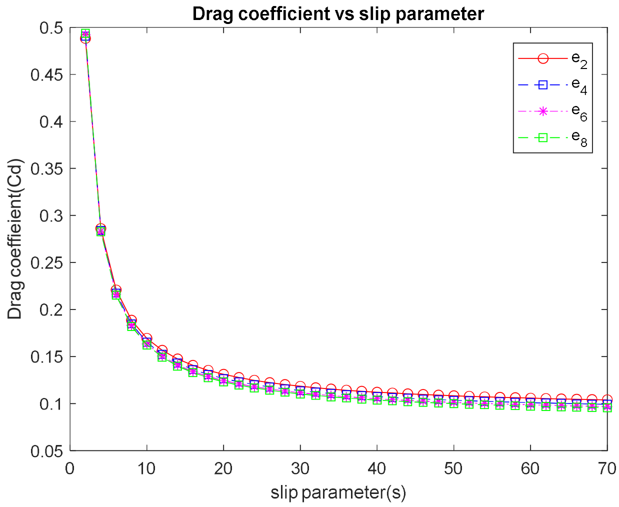


Figure 2 Drag coefficient(Cd) vs the slip parameter(s).

Figure 3 For a few values of slip coefficient (s), varying drag coefficient (Cd) for a few values of the ratio of couple stress parameter coefficient (e) are drawn at fixed μ=10. **Table 2** gives the numerical values.

Table 2 Drag coefficient (Cd) values for varying couple stress parameter coefficient (e) at different slip parameter (s).

SI.No	$s\backslash e$	$e=2$	$e=4$	$e=6$	$e=8$
1	$s=14$	0.2964	0.2937	0.2925	0.2918
2	$s=28$	0.2777	0.2746	0.2732	0.2724
3	$s=42$	0.2714	0.2683	0.2668	0.2659
4	$s=56$	0.2683	0.2651	0.2636	0.2627

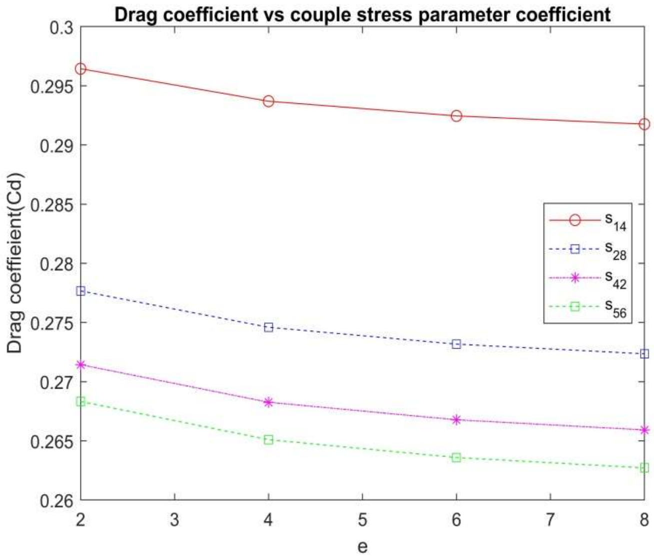
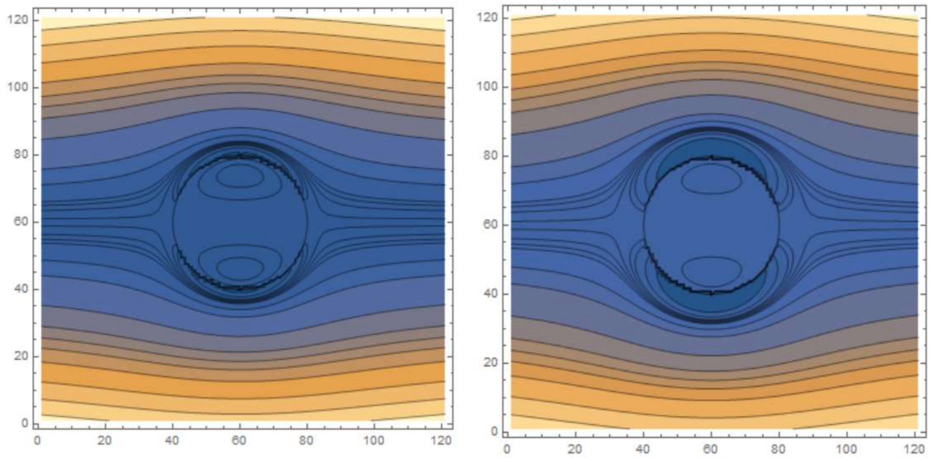


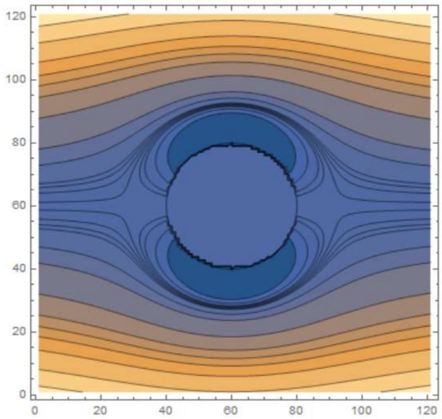
Figure 3 Drag coefficient(Cd) vs couple stress parameter coefficient(e).

It is observed that with a rise in the values of e, the ratio of couple stress parameters, the drag coefficient is falling.

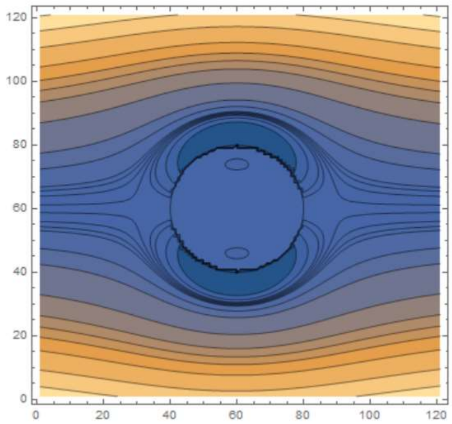
Figure 4 The flow pattern is presented with varying slip parameters at a fixed viscosity ratio, couple stress parameter. It was noticed that with a rise in the values of the slip parameter the external flow pattern is exhibiting more circulations with more area near the poles and internal flow is gradually disappearing.



(a) at fixed $\mu = 10, e = 5,\ s = 1$ (b) at fixed $\mu = 10, e = 5,\ s = 5$



(c) at fixed $\mu = 10,\ e = 5,\ s = 10$



(d) at fixed $\mu = 10, e = 5,\ s = 30$

Figure 4 The contour graphs are drawn at fixed viscosity coefficient (μ), couple stress parameter (e) and varying of slip parameter (s).

Conclusions

We considered a couple stress fluid flow past couple stress fluid sphere with interfacial slip condition. The velocity field is expressed with regard to the stream function. The drag coefficient (Cd) is evaluated. The results are obtained for the special cases in the limit form for the following cases

1) Results for the case of viscous fluid can be recovered as a limiting case of our analysis by taking $\lambda_e \to \infty$, $\lambda_i \to \infty$, and $\chi_1 \to \infty$, $\chi_2 \to \infty$.

2) Results of the no-slip condition, for the corresponding problem, can be captured as $s \to \infty$**.** Further, as $\mu \to \infty$, fluid sphere can reduce to the case of a solid sphere.

3) As the slip parameter raises the drag coefficient decreases. Further, the drag coefficient falls with the raise of the couple stress parameter.

The outcome for all the above special cases is in good accordance with the results reported by Happel and Brenner [13].

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References

[1] VK Stokes. Effects of couple stresses in fluids on the creeping flow past a sphere. *Phys. Fluid.* 1971; **14**, 1580-2.

[2] H Ramkissoon. Drag in couple stress fluids. *J. Appl. Math. Mech. Z. Angew. Math. Mech.* 1978; **29**, 341-6.

[3] TKV Iyengar and D Srinivasacharya. Stokes flow of an incompressible cosserat fluid past an approximate sphere. *Far East J. Math. Sci.* 1996; **4**, 283-97.

[4] P Aparna and JVR Murthy. Uniform flow of an incompressible couple stress fluid past a permeable sphere. *Int. Electron. Eng. Math. Soc.* 2010; **8**, 1-10.

[5] JC Umavathi, JP Kumar, JA Chamkha. Convective flow of two immiscible viscous and couple stress permeable fluids through a vertical channel. *Turk. J. Eng. Environ. Sci.* 2009; **33**, 221-43.

[6] TKV Iyengar and VG Vani. Flow of an incompressible couple stress fluid in the region between two slowly rotating confocal oblate spheroids. *Global J. Math. Sci.* 2012; **4**, 337-52.

[7] W Khan and F Yousafzai. On the exact solutions of couple stress fluids. *Adv. Trends Math.* 2014; **1**, 27-32.

[8] EA Ashmawy. Drag on a slip spherical particle moving in a couple stress fluid. *Alexandria Eng. J.* 2016; **55**, 1159-64.

[9] R Ellahi, A Zeeshan, F Hussain and A Asadollahi. Peristaltic blood flow of couple stress fluid suspended with nanoparticles under the influence of chemical reaction and activation energy. *Symmetry* 2019; **11**, 276.

[10] R Krishnan and P Shukla. Creeping flow of couple stress fluid past a fluid sphere with solid core. *J.*

Appl. Math. Mech. Z. Angew. Math. Mech. 2021; **101**, e202000115.

[11] NS Alsudais, S El-sapa and EA Ashmawy. Stokes flow of an incompressible couple stress fluid confined between two eccentric spheres. *Eur. J. Mech. B Fluid.* 2022; **91**, 244-52.

[12] AB Basset. *A treatise on hydrodynamics: With numerous examples*. Cambridge: Deighton Bell & Co, London, 1888.

[13] J Happel and H Brenner. *Low Reynolds number hydrodynamics*. Springer Dordrecht, Berlin, Germany, 1983.

[14] M Devakar, D Sreenivasu and B Shankar. Analytical solutions of couple stress fluid flows with slip boundary conditions. *Alexandria Eng. J.* 2014; **53**, 723-30.

[15] ZG Feng, EE Michaelides. A numerical study on the transient heat transfer from a sphere at high reynolds and peclet numbers. *Int. J. Heat Mass Tran.* 2000; **43**, 219-29.

[16] JVR Murthy and PK Meduri. Drag over Contaminated Fluid Sphere with Slip Condition. *Int. J. Sci. Eng. Res.* 2014; **5**, 719-27.

[17] JVR Murthy and MP Kumar. Exact solution for flow over a contaminated fluid sphere for stokes flow. *J. Phys. Conf.* 2015; **662**, 012016.