https://doi.org/10.48048/tis.2022.2499

Rotationally Symmetric Yang-Mills Gauge Theory and Generalized Uncertainty Principle

Atale Omprakash

Khandesh College Education Society's Moolji Jaitha College, Jalgaon, Maharashtra, India

(Corresponding author's e-mail: atale.om@outlook.com)

Received: 7 February 2022, Revised: 20 June 2022, Accepted: 27 June 2022, Published: 15 November 2022

Abstract

Recently, it has been of considerably high interest to introduce the notion of minimal length in quantum mechanics and quantum field theories to get a proper description of quantum gravity. In this paper, we have attempted to unify the notion of minimal length with $R \times S^3$ topological field theories which were introduced by Carmeli and Malin in 1985 [14]. We have collectively called this $(R \times S^3)_H$ topological field theories where the "H" in the sub-script stands for Heisenberg's notion of minimal length. A proper description for equations like Schrodinger's equation, Klein-Gordon equation and QED, QCD Lagrangian on $(R \times S^3)_H$ topology is derived.

Keywords: Quantum field theory, Minimal length, Schrodinger's equation, Dirac's equation, Klein-Gordon equation

Introduction

The holy grail of modern theoretical physics since mid-1900s is the problem of quantum gravity. Numerous attempts since then were done to unify quantum mechanics and general relativity. One interesting consequence of this unification is that in quantum gravity there exists a minimal observable distance at the order of Plank's length $l_p = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}$. It is widely assumed by physicists from last few decades that an existence of a minimal length is necessary for the consistent formulation of quantum gravity theories and it has been observed from the studies of blackhole that for any type of quantum gravity theory to exist, there should first exist a minimum measurable length in the theory that is near the magnitude of Plank's length. References for studying quantum theories at minimal length are [1-12]. Further exposure to the references regarding minimal length gravity theories alongside with quantum mechanics and quantum field theories can be found in [13]. The idea of the existence of a minimal length is not consistent with Heisenberg's uncertainty principle because according to it, the minimum measurable length is 0. This inconsistency is removed by introducing a parameter in the uncertainty principle and the resulting uncertainty principle is known as generalized uncertainty principle [7]. Since the equation for uncertainty principal changes, the mathematical definition of underlying operators also changes. The generalized uncertainty principle can be written as:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \beta \frac{\Delta p}{\hbar} \tag{1}$$

where β is the deformation parameter which determines the strength of the modification of the uncertainty principle and is of order of Plank's length. This leads to the following modification in the commutation relation between position and momentum operators:

$$[\hat{x}^\mu \hat{p}_\nu] = i\hbar\delta_{\mu\nu}\left(1 + \beta\hat{p}^\rho\hat{p}_\rho\right) \tag{2}$$

Operators in momentum space and position space now become [13] and,

$$\hat{x}^\mu = i\left(1 + \beta\hat{p}^\rho\hat{p}_\rho\right)\frac{\partial}{\partial p_\mu}, \qquad \hat{p}_\mu = p_\mu, \tag{3}$$

and

$$\hat{x}^{\mu} = x^{\mu}, \hat{p}_{\mu} = -i\hbar\left(1 - \beta\,\partial^{\rho}\,\partial_{\rho}\right)\partial_{\mu}, \tag{4}$$

respectively. The above representation of operators in fundamental in formulating theories at minimal length.

$R \times S^3$ topological field theory was established by Carmeli and Malka [20] a series of 7 papers [14-20]. This attempt was motivated to describe those systems which are solely angular dependent rather than distance as in Minkowskian space-time. In $R \times S^3$ topology, x is replaced with $\theta = (\theta^1, \theta^2, \theta^3)$ which are 3 rotational angles such as Euler angles and $L = (L_1, L_2, L_3) = \left(L_x, L_y, L_z\right)$ is the corresponding differential operator given by,

$$L_1 = \frac{-\sin\theta^3}{\sin\theta^2}\frac{\partial}{\partial\theta^1} - \cos\theta^3\frac{\partial}{\partial\theta^2} + \cot\theta^3\sin\theta^3\frac{\partial}{\partial\theta^3}, \tag{5}$$

$$L_2 = \frac{-\cos\theta^3}{\sin\theta^2}\frac{\partial}{\partial\theta^1} - \sin\theta^3\frac{\partial}{\partial\theta^2} + \cot\theta^3\cos\theta^3\frac{\partial}{\partial\theta^3}, \tag{6}$$

$$L_3 = -\frac{\partial}{\partial\theta^3}. \tag{7}$$

The operator $L^2 = L_1^2 + L_2^2 + L_3^2 = L_x^2 + L_y^2 + L_z^2$ is then give by,

$$L^2 = \frac{1}{\sin\theta^2}\frac{\partial}{\partial\theta^2}\left(\sin\theta^2\frac{\partial}{\partial\theta^2}\right) + \frac{1}{\sin^2\theta^2}\left(\frac{\partial^2}{\partial{\theta^1}^2} - 2\cos\theta^2\frac{\partial^2}{\partial\theta^1\,\partial\theta^3} + \frac{\partial^2}{\partial{\theta^3}^2}\right). \tag{8}$$

In relativistic notation $\theta^{\alpha} = (\theta^0, \theta^1, \theta^2, \theta^3)$, $\theta^0 = ct$ and,

$$J^{\alpha} = \left(J^0, J^k\right) = (E, \gamma J), \tag{9}$$

$$J_{\alpha} = (J_0, J_k) = (E, -\gamma J) \tag{10}$$

is the angular momentum four vector with $J = \left(J_x, J_y, J_z\right)$ and $J_k = i\hbar\gamma L_k$. Furthermore, using the definition of J^{α}, J_{α} and $\gamma = c(m_0/I_0)^{1/2}$ we can conclude the following relation:

$$E_J = J^0 = J_0 = \pm(\gamma^2 J^2 + I_0^2\gamma^4)^{1/2}, \tag{11}$$

$$J^{\alpha}J_{\alpha} = E_J^2 - \gamma^2 J^2 = I_0^2\gamma^4. \tag{12}$$

here, m_0 and I_0 are rest mass and moment of inertia, respectively.

There are 2 ways in which we could construct $(R \times S^3)_H$ topological field theories. We get the same results however the approaches differ on priority basis. When $R \times S^3$ topological field theories were constructed by Carmeli and Malka [20], they did so by making the transition,

$$p = -i\hbar\nabla \rightarrow J = -i\hbar L. \tag{13}$$

Thus, for example, the standard form of Schrodinger's equation in Cartesian coordinates was transformed into $R \times S^3$ topological form by simply making the transition from P to J and making the wave function depend on θ instead of x. Therefore, [14],

$$\left(\frac{-\hbar^2}{2m_0}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{14}$$

changes to

$$\left(\frac{-\hbar^2}{2I_0}L^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{15}$$

Now, the 2 methods that we mentioned are: First, writing down equations corresponding to minimal length and then making the transition to $R \times S^3$ topology. Second, making the transition to $R \times S^3$ topology and then formulating it in the framework of minimal length. We are going to follow the first route since most of the theories are already formulated in the framework of minimal length and now, we just have to make the transition to arrive on $(R \times S^3)_H$ topological field theories.

Schrodinger's equation and Klein-Gordon equation

The main aim of this section is to introduce Schrodinger's equation and Klein-Gordon equation on $(R \times S^3)_H$ topology. Since now we are dealing with coordinated containing Euler's angles, for the purpose of simplicity, we wouldn't be writing the whole equivalent definition of differential operator every time, rather, we would use the short hand notation. Now, Eqs. (3) and (4) in $R \times S^3$ topological coordinates takes the following form:

$$\hat{\theta}^{\mu} = i\left(1 + \beta \hat{J}^{\rho}\hat{J}_{\rho}\right)\frac{\partial}{\partial J_{\mu}}, \qquad \hat{J}_{\mu} = J_{\mu} \tag{16}$$

and

$$\hat{\theta}^{\mu} = \theta^{\mu}, \qquad \hat{J}_{\mu} = -i\hbar\left(1 - \beta L^{\rho}L_{\rho}\right)L_{\mu}, \tag{17}$$

respectively. There are 3 types of Schrodinger's equation that we are going to study: First and second equation are in coordinate space whereas the third equation is in momentum space.

Consider the following minimal length Schrodinger's equation from [21]:

$$\left(\frac{p_{op}^2}{2m_0} + V(x)\right)\psi_n(x) = E_n\psi_n(x) \tag{18}$$

where $p_{op} = p(1 + \beta p^2)$. If $E_n^{(0)}$ is the eigenvalue of

$$H_0 = \frac{p^2}{2m_0} + V(x) \tag{19}$$

then Eq. (18) can explicitly be written as

$$\frac{\hbar^2}{2m_0}\frac{d^2\psi_n(x)}{dx^2} + \left(E_n - 4m_0\beta\left(E_n^{(0)} - V(x)\right)^2 - V(x)\right)\psi_n(x) = 0. \tag{20}$$

This equation was originally introduced to calculate the scattering states of Woods-Saxon interaction. Now, using representation from Eq. (13), if $E_n^{(0)}$ is now the eigenvalue of

$$H_0 = \frac{J^2}{2I_0} + V(\theta) \tag{21}$$

then Eq. (20) on $(R \times S^3)_H$ can be written as

$$\frac{\hbar^2}{2I_0}L_x^2\psi_n(\theta) + \left(E_n - 4I_0\beta\left(E_n^{(0)} - V(\theta)\right)^2 - V(\theta)\right)\psi_n(\theta) = 0. \tag{22}$$

Authors in [22] came up with a minimal length Schrodinger's equation that can be solved by factorization method:

$$\frac{\partial^4\psi(x)}{\partial x^4} - \frac{3}{2\beta\hbar^2}\frac{\partial^2\psi(x)}{\partial x^2} - \frac{3m_0E\prime}{\beta\hbar^4}\psi(x) = 0 \tag{23}$$

The corresponding momentum operator that they have used is slightly different form usually what it is,

$$p_{op} = p\left(1 + \frac{\beta}{3}p^2\right). \tag{24}$$

On $(R \times S^3)_H$ topology, Eq. (23) can be written as

$$L_x^4\psi(x) - \frac{3}{2\beta\hbar^2}L_x^2\psi(x) - \frac{3I_0E'}{\beta\hbar^4}\psi(x) = 0. \tag{25}$$

Now let us focus our attention on stationary state minimal length Schrodinger's equation for harmonic oscillator in momentum space [1]:

$$\frac{d^2\psi(p)}{dp^2} + \frac{2\beta p}{1+\beta p^2}\frac{d\psi(p)}{dp} + \frac{1}{(1+\beta p^2)^2}(\varepsilon - \eta^2p^2)\psi(p) = 0 \tag{26}$$

where

$$\varepsilon = \frac{2E}{m_0\hbar^2\omega^2} \tag{27}$$

and

$$\eta = \frac{1}{(m_0\hbar\omega)^2}. \tag{28}$$

Corresponding Hamiltonian for above equation is

$$H = \frac{P^2}{2m_0} + m_0\omega^2\frac{x^2}{2}. \tag{29}$$

On $(R \times S^3)_H$ topology, Eq. (26) takes the form

$$\frac{d^2\psi(J)}{dJ^2} + \frac{2\beta J}{1+\beta J^2}\frac{d\psi(J)}{dJ} + \frac{1}{(1+\beta J^2)^2}(\varepsilon - \eta^2J^2)\psi(J) = 0 \tag{30}$$

where m_0 in Eq. (27) and (28) is now replaced by I_0 and the corresponding Hamiltonian is thus given by

$$H = \frac{J^2}{2I_0} + I_0\omega^2\frac{\theta^2}{2}. \tag{31}$$

We know that standard Klein-Gordon equation in terms of momentum can be written as $(\hbar = c = 1)$

$$\left(p^\mu p_\mu + m_0^2\right)\phi = 0 \tag{32}$$

For a simpler case of $1 + 1$ dimensions, consider the following form

$$(p_t^2 - p_x^2 + m_0^2)\phi = 0. \tag{33}$$

By the means of auxiliary variables [3], one can arrive at the following equation:

$$\left(\partial^\mu\partial_\mu + 2\beta\left(\partial^\mu\partial_\mu\right)\left(\partial^\mu\partial_\mu\right) + m_0^2\right)\phi = 0. \tag{34}$$

Thus, on $(R \times S^3)_H$ topology, one can write the Klein-Gordon equation as

$$\left(L^\mu L_\mu + 2\beta\left(L^\mu L_\mu\right)\left(L^\mu L_\mu\right) + I_0^2\right)\phi = 0. \tag{35}$$

Dirac's equation: Global, local and other complicated symmetries of the Lagrangian

Consider the standard Dirac's Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu + m\right)\psi. \tag{36}$$

The corresponding minimal length Dirac's Lagrangian thus can be given by [13]

$$\mathcal{L} = \overline{\psi}\big(\mathrm{i}\big(1 - \beta\,\partial^{\rho}\,\partial_{\rho}\big)\gamma^{\mu}\,\partial_{\mu} + \mathrm{m}\big)\psi \tag{37}$$

and thus, the corresponding $(\mathrm{R}\times \mathrm{S}^3)_{\mathrm{H}}$ topological Dirac's Lagrangian takes the following form [20,23]:

$$\mathcal{L} = \overline{\psi}\big(\mathrm{i}\big(1 - \beta\,\partial^{\rho}\,\partial_{\rho}\big)\gamma^{\mu}\,\partial_{\mu} + \mathrm{m}\big)\psi \tag{38}$$

$$= \overline{\psi}\,\tfrac{\mathrm{i}}{2}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi - \overline{\psi}\,\tfrac{\mathrm{i}}{2}\beta \mathrm{L}^{\rho}\mathrm{L}_{\rho}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi + \tfrac{3}{2\mathrm{a}}\overline{\psi}\gamma^{0}\gamma^{5}\psi - \mathrm{I}_0\overline{\psi}\psi. \tag{39}$$

here $\overline{\psi}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi = \overline{\psi}\big(\mathrm{L}_{\mu}\psi\big) - \big(\mathrm{L}_{\mu}\overline{\psi}\big)\psi$, I_0 is the rest momentum of inertia of the particle and the additional term has been added because such a term does not violate parity nor parity $+$ charge conjugation [23]. The constant a denotes radius of the universe. The additional term containing a gets smaller and smaller as time passes by but surely had played an important role in the beginning of the universe after the big bang.

The $(\mathrm{R}\times \mathrm{S}^3)_{\mathrm{H}}$ topological Dirac's Lagrangian can be seen to be invariant under global $\mathrm{U}(1)$ phase transformation. Thus, if we apply the transformations

$$\psi(\theta) \to \psi'(\theta) = \mathrm{e}^{-\mathrm{i}\xi}\psi(\theta) \tag{40}$$

$$\overline{\psi}(\theta) \to \overline{\psi'}(\theta) = \overline{\psi}(\theta)\mathrm{e}^{\mathrm{i}\xi} \tag{41}$$

or the infinitesimal form of it

$$\delta_{\varepsilon}\psi(\theta) = \psi'(\theta) - \psi(\theta) = -\mathrm{i}\varepsilon\psi(\theta) \tag{42}$$

$$\delta_{\varepsilon}\overline{\psi}(\theta) = \overline{\psi'}(\theta) - \overline{\psi}(\theta) = \mathrm{i}\varepsilon\overline{\psi}(\theta) \tag{43}$$

then Lagrangian (39) transforms as

$$\mathcal{L}' = \overline{\psi'}\,\tfrac{\mathrm{i}}{2}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi' - \overline{\psi'}\,\tfrac{\mathrm{i}}{2}\beta \mathrm{L}^{\rho}\mathrm{L}_{\rho}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi' + \tfrac{3}{2\mathrm{a}}\overline{\psi'}\gamma^{0}\gamma^{5}\psi' - \mathrm{I}_0\overline{\psi'}\psi' \tag{44}$$

$$= \overline{\psi}\mathrm{e}^{\mathrm{i}\xi}\,\tfrac{\mathrm{i}}{2}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\mathrm{e}^{-\mathrm{i}\xi}\psi - \overline{\psi}\mathrm{e}^{\mathrm{i}\xi}\,\tfrac{\mathrm{i}}{2}\beta \mathrm{L}^{\rho}\mathrm{L}_{\rho}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\mathrm{e}^{-\mathrm{i}\xi}\psi + \tfrac{3}{2\mathrm{a}}\overline{\psi}\mathrm{e}^{\mathrm{i}\xi}\gamma^{0}\gamma^{5}\mathrm{e}^{-\mathrm{i}\xi}\psi - \mathrm{I}_0\overline{\psi}\mathrm{e}^{\mathrm{i}\xi}\mathrm{e}^{-\mathrm{i}\xi}\psi \tag{45}$$

$$= \overline{\psi}\,\tfrac{\mathrm{i}}{2}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi - \overline{\psi}\,\tfrac{\mathrm{i}}{2}\beta \mathrm{L}^{\rho}\mathrm{L}_{\rho}\gamma^{\mu}\,\overleftrightarrow{\mathrm{L}_{\mu}}\psi + \tfrac{3}{2\mathrm{a}}\overline{\psi}\gamma^{0}\gamma^{5}\psi - \mathrm{I}_0\overline{\psi}\psi \tag{46}$$

$$= \mathcal{L} \tag{47}$$

This concludes that Lagrangian (39) is invariant under the global $\mathrm{U}(1)$ phase transformation. Now, we want to promote our global symmetry to local, namely

$$\delta_{\varepsilon}\psi(\theta) = -\mathrm{i}\varepsilon(\theta)\psi(\theta) \tag{48}$$

$$\delta_{\varepsilon}\overline{\psi}(\theta) = \mathrm{i}\varepsilon(\theta)\overline{\psi}(\theta) \tag{49}$$

and

$$\delta_{\varepsilon}\Big(\mathrm{L}_{\mu}\psi(\theta)\Big) = \mathrm{L}_{\mu}\big(\delta_{\varepsilon}\psi(\theta)\big) = \mathrm{L}_{\mu}\big(-\mathrm{i}\varepsilon(\theta)\psi(\theta)\big) \tag{50}$$

$$= -\mathrm{i}\Big(\Big(\mathrm{L}_{\mu}\varepsilon(\theta)\Big) + \varepsilon(\theta)\mathrm{L}_{\mu}\Big)\psi(\theta). \tag{51}$$

Things get a little bit tricky here due to the presence of the term $\overleftrightarrow{\mathrm{L}_{\mu}}$ in the Lagrangian. Therefore, it's perhaps better to separate those terms in Lagrangian which will change under variation from those which will remain same. Thus, we can write Lagrangian (Eq. (39)) as,

$$\mathcal{L} = \bar{\psi}\frac{i}{2}\left(1-\beta L^{\rho}L_{\rho}\right)\gamma^{\mu}\overleftrightarrow{L_{\mu}}\psi + \frac{3}{2a}\bar{\psi}\gamma^{0}\gamma^{5}\psi - I_{0}\bar{\psi}\psi \tag{52}$$

$$= \frac{i}{2}\left(1-\beta L^{\rho}L_{\rho}\right)\gamma^{\mu}\left(\bar{\psi}\left(L_{\mu}\psi\right) - \left(L_{\mu}\bar{\psi}\right)\psi\right) + \frac{3}{2a}\bar{\psi}\gamma^{0}\gamma^{5}\psi - I_{0}\bar{\psi}\psi \tag{53}$$

and the Lagrangian now transforms as,

$$\delta\mathcal{L} = \frac{i}{2}\left(1-\beta L^{\rho}L_{\rho}\right)\gamma^{\mu}\left(\delta\bar{\psi}\left(L_{\mu}\psi\right) + \bar{\psi}\left(L_{\mu}\delta\psi\right) - \left(L_{\mu}\delta\bar{\psi}\right)\psi - \left(L_{\mu}\bar{\psi}\right)\delta\psi\right) + \frac{3}{2a}\left(\delta\bar{\psi}\gamma^{0}\gamma^{5}\psi + \bar{\psi}\gamma^{0}\gamma^{5}\delta\psi\right) - I_{0}\delta\bar{\psi}\psi - I_{0}\bar{\psi}\delta\psi \tag{54}$$

$$= \frac{i}{2}\left(1-\beta L^{\rho}L_{\rho}\right)\gamma^{\mu}\left(i\varepsilon\bar{\psi}\left(L_{\mu}\psi\right) - i\bar{\psi}\left(L_{\mu}\varepsilon\psi + \varepsilon L_{\mu}\psi\right) - i\left(L_{\mu}\varepsilon\bar{\psi} + \varepsilon L_{\mu}\bar{\psi}\right)\psi + \left(L_{\mu}\bar{\psi}\right)i\varepsilon\psi\right) + \frac{3}{2a}\left(i\varepsilon\bar{\psi}\gamma^{0}\gamma^{5}\psi - \bar{\psi}\gamma^{0}\gamma^{5}i\varepsilon\psi\right) - I_{0}i\varepsilon\bar{\psi}\psi + I_{0}\bar{\psi}i\varepsilon\psi \tag{55}$$

$$= \frac{i}{2}\left(1-\beta L^{\rho}L_{\rho}\right)\gamma^{\mu}\left(i\varepsilon\bar{\psi}\left(L_{\mu}\psi\right) - i\bar{\psi}L_{\mu}\varepsilon\psi - i\bar{\psi}\varepsilon L_{\mu}\psi - iL_{\mu}\varepsilon\bar{\psi}\psi - i\varepsilon L_{\mu}\bar{\psi}\psi + \left(L_{\mu}\bar{\psi}\right)i\varepsilon\psi\right) \tag{56}$$

$$= -\left(1-\beta L^{\rho}L_{\rho}\right)\left(\gamma^{\mu}\bar{\psi}L_{\mu}\varepsilon\psi\right) \tag{57}$$

$$\neq 0 \tag{58}$$

Thus, Lagrangian (Eq. (39)) is not invariant under the local phase transformation because there is no additional term whose variation will cancel out the product that we have got. To make the Lagrangian invariant under the local phase transformation, we introduce the following covariant derivative:

$$D_{\mu}\psi(\theta) = \left(L_{\mu} + ieA_{\mu}(\theta)\right)\psi(\theta) \tag{59}$$

whose variation can be written as

$$\delta\left(D_{\mu}\psi(\theta)\right) = -i\varepsilon(\theta)D_{\mu}\psi(\theta) \tag{60}$$

and for the additional field A_{μ}

$$\delta\left(A_{\mu}(\theta)\right) = \frac{1}{e}L_{\mu}\varepsilon(\theta). \tag{61}$$

Therefore, we have

$$\mathcal{L} = \frac{i}{2}\left(1-\beta D^{\rho}D_{\rho}\right)\gamma^{\mu}\left(\bar{\psi}\left(D_{\mu}\psi\right) - \left(D_{\mu}\bar{\psi}\right)\psi\right) + \frac{3}{2a}\bar{\psi}\gamma^{0}\gamma^{5}\psi - I_{0}\bar{\psi}\psi \tag{62}$$

$$= \bar{\psi}\frac{i}{2}\left(1-\beta D^{\rho}D_{\rho}\right)\gamma^{\mu}\overleftrightarrow{D_{\mu}}\psi + \frac{3}{2a}\bar{\psi}\gamma^{0}\gamma^{5}\psi - I_{0}\bar{\psi}\psi \tag{63}$$

After substituting the value of covariant derivative and opening the brackets, Eq. (63) can explicitly be written as

$$\mathcal{L} = \frac{i}{2}(\gamma^{\mu}\bar{\psi}L_{\mu}\psi - \gamma^{\mu}\bar{\psi}ieA_{\mu}\psi - \gamma^{\mu}L_{\mu}\bar{\psi}\psi + \gamma^{\mu}ieA_{\mu}\bar{\psi}\psi - \beta L^{\rho}L_{\rho}\gamma^{\mu}\bar{\psi}L_{\mu}\psi + \beta L^{\rho}L_{\rho}\gamma^{\mu}\bar{\psi}ieA_{\mu}\psi + \beta L^{\rho}L_{\rho}\gamma^{\mu}L_{\mu}\bar{\psi}\psi - \beta L^{\rho}L_{\rho}\gamma^{\mu}ieA_{\mu}\bar{\psi}\psi + \beta L^{\rho}ieA_{\rho}\gamma^{\mu}\bar{\psi}L_{\mu}\psi - \beta L^{\rho}ieA_{\rho}\gamma^{\mu}\bar{\psi}ieA_{\mu}\psi - \beta L^{\rho}ieA_{\rho}\gamma^{\mu}L_{\mu}\bar{\psi}\psi + \beta L^{\rho}ieA_{\rho}\gamma^{\mu}ieA_{\mu}\bar{\psi}\psi + \beta ieA^{\rho}L_{\rho}\gamma^{\mu}\bar{\psi}L_{\mu}\psi - \beta ieA^{\rho}L_{\rho}\gamma^{\mu}\bar{\psi}ieA_{\mu}\psi - \beta ieA^{\rho}L_{\rho}\gamma^{\mu}L_{\mu}\bar{\psi}\psi + \beta ieA^{\rho}L_{\rho}\gamma^{\mu}ieA_{\mu}\bar{\psi}\psi - \beta ieA^{\rho}ieA_{\rho}\gamma^{\mu}\bar{\psi}L_{\mu}\psi + \beta ieA^{\rho}ieA_{\rho}\gamma^{\mu}\bar{\psi}ieA_{\mu}\psi + \beta ieA^{\rho}ieA_{\rho}\gamma^{\mu}L_{\mu}\bar{\psi}\psi - \beta ieA^{\rho}ieA_{\rho}\gamma^{\mu}ieA_{\mu}\bar{\psi}\psi) + \frac{3}{2a}\bar{\psi}\gamma^{0}\gamma^{5}\psi - I_{0}\bar{\psi}\psi. \tag{64}$$

One can see that the Lagrangian containing covariant derivatives and additional fields is now invariant under the local phase transformation.

Now, let us generalize our Lagrangian to more complicated symmetries. Consider the following Lagrangian containing several complex fields where $\overline{\psi}_k$ now belongs to a non trivial representation R of some internal symmetry group G:

$$\mathcal{L} = \overline{\psi}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\psi_k + \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5\psi_k - I_0\overline{\psi}_k\psi_k \tag{65}$$

where $k = 1,2,3,\ldots \mathrm{dimR}$ is the internal symmetry index. If we apply the transformations

$$\psi_k \to \psi'_k = (U\psi)_k = \left(e^{-i\theta^a T^a}\psi\right)_k \tag{66}$$

$$\overline{\psi}_k \to \overline{\psi'}_k = (\overline{\psi}U^\dagger)_k = \left(\overline{\psi}e^{-i\theta^a T^a}\right)_k \tag{67}$$

or the infinitesimal form of it

$$\delta_\varepsilon\psi_k = -i\varepsilon^a T^a_{kl}\psi_l \tag{68}$$

$$\delta_\varepsilon\overline{\psi}_k = i\varepsilon^a\overline{\psi}_l(T^a)_{lk} \tag{69}$$

where $a = 1,2,3\ldots \mathrm{dimG}$, ε^a and θ^a are infinitesimal and T^a (Hermitian) is the generator of internal symmetry group G satisfying Lie algebra of Lie group G:

$$\left[T^a, T^b\right] = if^{abc}T^c, \qquad a, b, c = \mathrm{dimG}. \tag{70}$$

Thus

$$\mathcal{L}' = \overline{\psi'}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\psi'_k + \frac{3}{2a}\overline{\psi'}_k\gamma^0\gamma^5\psi'_k - I_0\overline{\psi'}_k\psi'_k \tag{71}$$

$$= (\overline{\psi}U^\dagger)_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}(U\psi)_k + \frac{3}{2a}(\overline{\psi}U^\dagger)_k\gamma^0\gamma^5(U\psi)_k - I_0(\overline{\psi}U^\dagger)_k(U\psi)_k \tag{72}$$

$$= \left(\overline{\psi}e^{-i\theta^a T^a}\right)_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\left(e^{-i\theta^a T^a}\psi\right)_k + \frac{3}{2a}\left(\overline{\psi}e^{-i\theta^a T^a}\right)_k\gamma^0\gamma^5\left(e^{-i\theta^a T^a}\psi\right)_k - I_0\left(\overline{\psi}e^{-i\theta^a T^a}\right)_k\left(e^{-i\theta^a T^a}\psi\right)_k \tag{73}$$

$$= \overline{\psi}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\psi_k + \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5\psi_k - I_0\overline{\psi}_k\psi_k \tag{74}$$

$$= \mathcal{L} \tag{75}$$

Or the other way around, using Eqs. (68) and (69), we have

$$\delta\mathcal{L} = \delta\overline{\psi}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\psi_k + \overline{\psi}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \leftrightarrow L_\mu\delta\psi_k + \frac{3}{2a}\delta\overline{\psi}_k\gamma^0\gamma^5\psi_k + \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5\delta\psi_k - I_0\delta\overline{\psi}_k\psi_k - I_0\overline{\psi}_k\delta\psi_k \tag{76}$$

$$= i\varepsilon^a\overline{\psi}_l(T^a)_{lk} \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \overleftrightarrow{L_\mu}\psi_k - \overline{\psi}_k \frac{i}{2}\left(1 - \beta L^\rho L_\rho\right)\gamma^\mu \leftrightarrow L_\mu i\varepsilon^a T^a_{kl}\psi_l + \frac{3}{2a}i\varepsilon^a\overline{\psi}_l(T^a)_{lk}\gamma^0\gamma^5\psi_k - \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5 i\varepsilon^a T^a_{kl}\psi_l - I_0 i\varepsilon^a\overline{\psi}_l(T^a)_{lk}\psi_k + I_0\overline{\psi}_k i\varepsilon^a T^a_{kl}\psi_l \tag{77}$$

$$= 0 \tag{78}$$

where we have used the fact that $(T^a)^\dagger = T^a$ and that internal symmetry generators commute with gamma matrices. Let us now promote our global symmetry to local. For this, we apply the following transformations

$$\delta_\varepsilon\psi_k = -i\varepsilon^a(\theta)T^a_{kl}\psi_l \tag{79}$$

$$\delta_\varepsilon\overline{\psi}_k = i\varepsilon^a(\theta)\overline{\psi}_l T^a_{lk} \tag{80}$$

$$\delta\left(L_\mu\psi_k\right) = L_\mu\left(-i\varepsilon^a(\theta)T^a_{kl}\psi_l\right) = -i\left(L_\mu\varepsilon^a(\theta)\right)T^a_{kl}\psi_l - i\varepsilon^a(\theta)T^a_{kl}L_\mu\psi_l \tag{81}$$

$$\delta\left(L_\mu\bar{\psi}_k\right) = L_\mu\left(i\varepsilon^a(\theta)T^a_{lk}\bar{\psi}_l\right) = i\left(L_\mu\varepsilon^a(\theta)\right)T^a_{lk}\bar{\psi}_l + i\varepsilon^a(\theta)T^a_{lk}L_\mu\bar{\psi}_l \tag{82}$$

Thus, under the above local transformation, Lagrangian (Eq. (64)) transforms as

$$\delta\mathcal{L} = \frac{i}{2}\left(1-\beta L^\rho L_\rho\right)\gamma^\mu\left(\delta\bar{\psi}_k\left(L_\mu\psi_k\right) + \bar{\psi}_k\left(L_\mu\delta\psi_k\right) - \left(L_\mu\delta\bar{\psi}_k\right)\psi_k - \left(L_\mu\bar{\psi}_k\right)\delta\psi_k\right) + \frac{3}{2a}\left(\delta\bar{\psi}_k\gamma^0\gamma^5\psi_k + \bar{\psi}_k\gamma^0\gamma^5\delta\psi_k\right) - I_0\delta\bar{\psi}_k\psi_k - I_0\bar{\psi}_k\delta\psi_k$$

$$= \frac{i}{2}\left(1-\beta L^\rho L_\rho\right)\gamma^\mu\left(i\varepsilon^a\bar{\psi}_l T^a_{lk}\left(L_\mu\psi_k\right) - \bar{\psi}_k iL_\mu\varepsilon^a T^a_{kl}\psi_l - \bar{\psi}_k i\varepsilon^a T^a_{kl}L_\mu\psi_l - iL_\mu\varepsilon^a T^a_{lk}\bar{\psi}_l\psi_k - i\varepsilon^a T^a_{lk}L_\mu\bar{\psi}_l\psi_k + i\left(L_\mu\bar{\psi}_k\right)\varepsilon^a T^a_{kl}\psi_l\right) + \frac{3}{2a}\left(i\varepsilon^a(\theta)\bar{\psi}_l T^a_{lk}\gamma^0\gamma^5\psi_k - \bar{\psi}_k\gamma^0\gamma^5 i\varepsilon^a(\theta)T^a_{kl}\psi_l\right) - I_0 i\varepsilon^a(\theta)\bar{\psi}_l T^a_{lk}\psi_k + I_0\bar{\psi}_k i\varepsilon^a(\theta)T^a_{kl}\psi_l \tag{83}$$

$$= \frac{1}{2}\left(1-\beta L^\rho L_\rho\right)\gamma^\mu\left(\bar{\psi}_k L_\mu\varepsilon^a T^a_{kl}\psi_l + L_\mu\varepsilon^a T^a_{lk}\bar{\psi}_l\psi_k\right) \tag{84}$$

$$\neq 0. \tag{85}$$

Thus, as in the case of (53), there is no extra term here whose variation would cancel the extra term that we have obtained. Therefore, we now introduce the following covariant derivative with an appropriate coupling constant g:

$$D_\mu\psi = \left(L_\mu + igT^a A^a_\mu\right)\psi \tag{86}$$

whose variation is given by

$$\delta\left(D_\mu\psi\right)_k = -i\varepsilon^a(\theta)T^a_{kl}\left(D_\mu\psi\right)_l. \tag{87}$$

We can further write (86) as

$$D_\mu\psi_k = L_\mu\psi_k + igT^a_{kl}A^a_\mu\psi_l = \left(L_\mu\delta_{kl} + igT^a_{kl}A^a_\mu\right)\psi_l. \tag{88}$$

Thus

$$igT^a_{kl}\delta\left(A^a_\mu\psi_l\right) = \delta\left(D_\mu\psi_k\right) - L_\mu\delta\psi_k \tag{89}$$

$$= -i\varepsilon^a T^a_{kl}\left(D_\mu\psi\right)_l - L_\mu\delta\psi_k \tag{90}$$

$$= -i\varepsilon^a T^a_{kl}\delta\left(L_\mu\psi_l + igT^b_{lm}A^b_\mu\psi_m\right) - L_\mu\left(-i\varepsilon^a T^a_{kl}\psi_l\right) \tag{91}$$

$$= -i\varepsilon^a T^a_{kl}\delta\left(L_\mu\psi_l + igT^b_{lm}A^b_\mu\psi_m\right) + i\left(L_\mu\varepsilon^a\right)T^a_{kl}\psi_l + i\varepsilon^a T^a_{kl}L_\mu\psi_l \tag{92}$$

$$= i\left(L_\mu\varepsilon^a\right)T^a_{kl}\psi_l + g\varepsilon^a T^a_{kl}T^b_{lm}A^b_\mu\psi_m. \tag{93}$$

On the other hand, we can write $igT^a_{kl}\delta\left(A^a_\mu\psi_l\right)$ as

$$igT^a_{kl}\delta\left(A^a_\mu\psi_l\right) = igT^a_{kl}\delta A^a_\mu\psi_l + gT^a_{kl}A^a_\mu\delta\psi_l \tag{94}$$

$$= igT^a_{kl}\delta A^a_\mu\psi_l + gT^a_{kl}A^a_\mu\left(-i\varepsilon^b\right)T^b_{lm}\psi_m \tag{95}$$

$$= igT^a_{kl}\delta A^a_\mu\psi_l + g\varepsilon^a T^b_{kl}A^b_\mu T^a_{lm}\psi_m. \tag{96}$$

Substituting the above result in (89), we get

$$\mathrm{ig}T^a_{kl}\delta\left(A^a_\mu\psi_l\right) = \mathrm{i}\left(L_\mu\varepsilon^a\right)T^a_{kl}\psi_l + \mathrm{g}\varepsilon^a\left(T^aT^b\right)_{km}A^b_\mu\psi_m - \mathrm{g}\varepsilon^a\left(T^bT^a\right)_{km}A^b_\mu\psi_m \tag{97}$$

$$= \mathrm{i}\left(L_\mu\varepsilon^a\right)T^a_{kl}\psi_l + \mathrm{g}\varepsilon^a\left[T^a, T^b\right]_{km}A^b_\mu\psi_m \tag{98}$$

$$= \mathrm{i}\left(L_\mu\varepsilon^a\right)T^a_{kl}\psi_l + \mathrm{g}\varepsilon^a f^{abc}(T^c)_{km}A^b_\mu\psi_m. \tag{99}$$

Thus, after further simplification one can obtain the following variation:

$$\delta A^a_\mu = \frac{1}{g}L_\mu\varepsilon^a - f^{abc}A^b_\mu\varepsilon^c. \tag{100}$$

Therefore, using

$$\delta_\varepsilon\psi_k = -\mathrm{i}\varepsilon^a(\theta)T^a_{kl}\psi_l \tag{101}$$

$$\delta_\varepsilon\overline{\psi}_k = \mathrm{i}\varepsilon^a(\theta)\overline{\psi}_l T^a_{lk} \tag{102}$$

and Eq. (97), one can easily arrive at the conclusion that the Lagrangian

$$\mathcal{L} = \overline{\psi}_k\frac{\mathrm{i}}{2}\left(1 - \beta D^\rho D_\rho\right)\gamma^\mu\ \overleftrightarrow{D_\mu}\psi_k + \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5\psi_k - I_0\overline{\psi}_k\psi_k \tag{103}$$

is now invariant under the infinitesimal local phase transformation.

Electrodynamics and gauge fields

In this section, we are going to introduce the field strength tensor on $(R \times S^3)_H$ topology and then we will expose it to different transformations and symmetries as we did in previous case for our modified Dirac's Lagrangian in previous section. This will be a relatively short section as compare to the previous one because as we have now been already introduced to the concept of global symmetries, local symmetries and covariant derivative, we are going to omit obvious steps during calculations. Beginning from the field strength tensor on $(R \times S^3)$ topology, we have [17],

$$F_{\mu\nu} = L_\mu A_\nu - L_\nu A_\mu. \tag{104}$$

Now, the minimal length field strength tensor is given by [13],

$$\mathcal{F}_{\mu\nu} = -\mathrm{i}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] = -\mathrm{i}\left[\left(1 - \beta D^\rho D_\rho\right)D_\mu, \left(1 - \beta D^\rho D_\rho\right)D_\nu\right]. \tag{105}$$

Thus, we get

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - 2\beta D^\rho D_\rho F_{\mu\nu} - \beta\left(D^\rho F_{\mu\rho}D_\nu - D^\rho F_{\nu\rho}D_\mu\right) - \beta(F_{\mu\rho}D^\rho D_\nu - F_{\nu\rho}D^\rho D_\mu) + O(\beta^2) \tag{106}$$

and

$$\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = F_{\mu\nu}F^{\mu\nu} - 4\beta F_{\mu\nu}D^\rho D_\rho F^{\mu\nu} - 4\beta F_{\mu\nu}F^{\mu\rho}D_\rho D^\nu - 4\beta F_{\mu\nu}D_\rho F^{\mu\rho}D^\nu + O(\beta^2) \tag{107}$$

where the field strength tensor in the above equation is now defined by (104). Inserting the expression for covariant derivative yields

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} = {} & F_{\mu\nu}F^{\mu\nu} - \beta(4F_{\mu\nu}L_\rho L^\rho F^{\mu\nu} + 4\mathrm{i}L_\rho eA^\rho F_{\mu\nu}F^{\mu\nu} + 8\mathrm{i}F_{\mu\nu}eA_\rho L^\rho F^{\mu\nu} + 8\mathrm{i}L^\rho eA_\nu F_{\mu\rho}F^{\mu\nu} + \\ & 4\mathrm{i}eA^\rho F_{\mu\rho}L_\nu F^{\mu\nu} - 8eA^\rho F_{\mu\rho}eA_\nu F^{\mu\nu} - 4eA^\rho eA_\rho F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{108}$$

and further inserting the value of field strength tensor yields

$$F_{\mu\nu}F^{\mu\nu} = L_\mu A_\nu L^\mu A^\nu - L_\mu A_\nu L^\nu A^\mu - L_\nu A_\mu L^\mu A^\nu + L_\nu A_\mu L^\nu A^\mu - 4\beta L_\mu A_\nu L_\rho L^\rho L^\mu A^\nu + 4\beta L_\nu A_\mu L_\rho L^\rho L^\mu A^\nu + 4\beta L_\mu A_\nu L_\rho L^\rho L^\nu A^\mu - 4\beta L_\nu A_\mu L_\rho L^\rho L^\nu A^\mu - 4\beta i L_\rho e A^\rho L_\mu A_\nu L^\mu A^\nu + 4\beta i L_\rho e A^\rho L_\mu A_\nu L^\nu A^\mu + 4\beta i L_\rho e A^\rho L_\nu A_\mu L^\mu A^\nu - 4\beta i L_\rho e A^\rho L_\nu A_\mu L^\nu A^\mu - 8\beta i L_\mu A_\nu e A_\rho L^\rho L^\mu A^\nu + 8\beta i L_\nu A_\mu e A_\rho L^\rho L^\mu A^\nu + 8\beta i L_\mu A_\nu e A_\rho L^\rho L^\nu A^\mu - 8\beta i L_\nu A_\mu e A_\rho L^\rho L^\nu A^\mu - 8\beta i L^\rho e A_\nu L_\mu A_\rho L^\mu A^\nu + 8\beta i L^\rho e A_\nu L_\rho A_\mu L^\mu A^\nu + 8\beta i L^\rho e A_\nu L_\mu A_\rho L^\nu A^\mu - 8\beta i L^\rho e A_\nu L_\rho A_\mu L^\nu A^\mu - 4\beta i e A^\rho L_\mu A_\rho L_\nu L^\mu A^\nu + 4\beta i e A^\rho L_\rho A_\mu L_\nu L^\mu A^\nu + 4\beta i e A^\rho L_\mu A_\rho L_\nu L^\nu A^\mu - 4\beta i e A^\rho L_\rho A_\mu L_\nu L^\nu A^\mu + 8\beta e A^\rho L_\mu A_\rho e A_\nu L^\mu A^\nu - 8\beta e A^\rho L_\rho A_\mu e A_\nu L^\mu A^\nu - 8\beta e A^\rho L_\mu A_\rho e A_\nu L^\nu A^\mu + 8\beta e A^\rho L_\rho A_\mu e A_\nu L^\nu A^\mu + 4\beta e A^\rho e A_\rho L_\mu A_\nu L^\mu A^\nu - 4\beta e A^\rho e A_\rho L_\mu A_\nu L^\nu A^\mu - 4\beta e A^\rho e A_\rho L_\nu A_\mu L^\mu A^\nu + 4\beta e A^\rho e A_\rho L_\nu A_\mu L^\nu A^\mu. \tag{109}$$

Note that the above representation of the field strength tensor on $(R \times S^3)_H$ topology is invariant under the infinitesimal local phase transformation. Using the above expression, we can construct the dynamical part of the gauge field which when added to Dirac's Lagrangian would give a complete Lagrangian for quantum electrodynamics. For defining the finite symmetry transformation matrix, we can define a unitary representation of the group. Thus, let

$$U = e^{-i\theta^a(\theta)T^a} \tag{110}$$

and define the following set of transformations

$$\psi \to U\psi \tag{111}$$

$$\bar{\psi} \to \bar{\psi}U \tag{112}$$

$$A_\mu = UA_\mu U^{-1} - \frac{1}{ig}\left(L_\mu U\right)U^{-1}. \tag{113}$$

where $A_\mu = T^a A_\mu^a$. One can see that $F_{\mu\nu}$ is not invariant under the above set of transformations:

$$F_{\mu\nu} = \frac{1}{ig}\left(L_\mu U L_\nu U^{-1} - L_\nu U L_\mu U^{-1}\right) + L_\mu U A_\nu U^{-1} + U A_\nu L_\mu U^{-1} - L_\nu U A_\mu U^{-1} - U A_\mu L_\nu U^{-1} + U\left(L_\mu A_\nu - L_\nu A_\mu\right)U^{-1}. \tag{114}$$

Note that under the gauge transformation, $ig\left[A_\mu, A_\nu\right]$ transforms as

$$ig\left[A_\mu, A_\nu\right] = igU\left[A_\mu, A_\nu\right]U^{-1} - \frac{1}{ig}\left(L_\mu U L_\nu U^{-1} - L_\nu U L_\mu U^{-1}\right) - \left[L_\mu U A_\nu U^{-1} + U A_\nu L_\mu U^{-1} - L_\nu U A_\mu U^{-1} - U A_\mu L_\nu U^{-1}\right] \tag{115}$$

Comparing (114) and (115) we can see that if we write field strength tensor as,

$$F_{\mu\nu} = L_\mu A_\nu - L_\nu A_\mu + ig\left[A_\mu, A_\nu\right] \tag{116}$$

then under that gauge transformations (111) - (113) $F_{\mu\nu}$ transforms covariantly:

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1}. \tag{117}$$

In component form, writing A_μ as $T^a A_\mu^a$, (116) takes the form

$$F_{\mu\nu}^a = L_\mu A_\nu^a - L_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c = -F_{\nu\mu}^a. \tag{118}$$

and thus

$$F_{\mu\nu}^a F^{\mu\nu a} = \left(L_\mu A_\nu^a - L_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c\right)(L^\mu A^{\nu a} - L^\nu A^{\mu a} - gf^{apq}A^{\mu p}A^{\nu q}). \tag{119}$$

Conclusions

In this paper, we have established $(R \times S^3)_H$ topological field theories where we have unified the notion of minimal length in field theories on $(R \times S^3)$topology. In the beginning we derive a set of Schrodinger's equations along with Klein-Gordon equation. From the above results, we can now construct quantum electrodynamics Lagrangian on $(R \times S^3)_H$ as

$$\mathcal{L}_{QED} = \overline{\psi}\frac{i}{2}\left(1 - \beta D^\rho D_\rho\right)\gamma^\mu \overleftrightarrow{D_\mu}\psi + \frac{3}{2a}\overline{\psi}\gamma^0\gamma^5\psi - I_0\overline{\psi}\psi - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu} - \beta(4F_{\mu\nu}L_\rho L^\rho F^{\mu\nu} + 4iL_\rho eA^\rho F_{\mu\nu}F^{\mu\nu} + 8iF_{\mu\nu}eA_\rho L^\rho F^{\mu\nu} + 8iL^\rho eA_\nu F_{\mu\rho}F^{\mu\nu} + 4ieA^\rho F_{\mu\rho}L_\nu F^{\mu\nu} - 8eA^\rho F_{\mu\rho}eA_\nu F^{\mu\nu} - 4eA^\rho eA_\rho F_{\mu\nu}F^{\mu\nu}) \tag{120}$$

The above Lagrangian is invariant under the infinitesimal local phase transformation:

$$\delta_\varepsilon\psi(\theta) = -i\varepsilon(\theta)\psi(\theta) \tag{121}$$

$$\delta_\varepsilon\overline{\psi}(\theta) = i\varepsilon(\theta)\overline{\psi}(\theta) \tag{122}$$

and

$$\delta\left(A_\mu(\theta)\right) = \frac{1}{e}L_\mu\varepsilon(\theta). \tag{123}$$

Similarly, we can construct quantum chromodynamics Lagrangian which will describe fermions integrating with a non-Abelian gauge field on $(R \times S^3)_H$ topology. Thus

$$\mathcal{L}_{QCD} = \overline{\psi}_k\frac{i}{2}\left(1 - \beta D^\rho D_\rho\right)\gamma^\mu \overleftrightarrow{D_\mu}\psi_k + \frac{3}{2a}\overline{\psi}_k\gamma^0\gamma^5\psi_k - I_0\overline{\psi}_k\psi_k - \frac{1}{4}(F^a_{\mu\nu}F^{\mu\nu a} - \beta(4F^a_{\mu\nu}L_\rho L^\rho F^{\mu\nu a} + 4iL_\rho eA^\rho F^a_{\mu\nu}F^{\mu\nu a} + 8iF^a_{\mu\nu}eA_\rho L^\rho F^{\mu\nu a} + 8iL^\rho eA_\nu F^a_{\mu\rho}F^{\mu\nu a} + 4ieA^\rho F^a_{\mu\rho}L_\nu F^{\mu\nu a} - 8eA^\rho F^a_{\mu\rho}eA_\nu F^{\mu\nu a} - 4eA^\rho eA_\rho F^a_{\mu\nu}F^{\mu\nu a}). \tag{124}$$

This Lagrangian is invariant under the local phase transformation:

$$\psi \rightarrow U\psi \tag{125}$$

$$\overline{\psi} \rightarrow \overline{\psi}U \tag{126}$$

$$A_\mu = UA_\mu U^{-1} - \frac{1}{ig}\left(L_\mu U\right)U^{-1}. \tag{127}$$

or the infinitesimal form of it. Here $A_\mu = T^a A^a_\mu$. The above presented model is based on a simple generalized uncertainty principle. There exist many other generalized formulations of the uncertainty principle constructed for different purposes.

The main reason of formulating field theories on $R \times S^3$ topology is due to the fact that the surface of the sphere S^3 in a space has considerably more symmetries than other spaces. From the results that we have presented in this paper, it would be interesting to canonically quantize the above lightly presented version of Yang-Mills theory and introduce path integral formulation which would indeed help us to introduce Faddeev-Popov ghosts and BRST invariance in our theory.

References

[1] A Kempf, G Mangano and RB Mann. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev. D* 1995; **52**, 1108.

[2] A Awad and A Ali. Planck-scale corrections to Friedmann equation. *Cent. Eur. J. Phys.* 2014; **12**, 245-5.

[3] P Bosso, S Das and V Todorinov. Quantum field theory with the generalized uncertainty principle I: Scalar electrodynamics. *Ann. Phys.* 2020; **422**, 168319.

[4] P Bosso, S Das and V Todorinov. Quantum field theory with the generalized uncertainty principle II: Quantum leectrodynamics. *Ann. Phys.* 2020; **424**, 168350.

[5] LN Chang, Z Lewis, D Minic and T Takeuchi. On the minimal length uncertainty relation and the foundations of string theory. *Adv. High Energ. Phys.* 2011; **2011**, 493514.

[6] H Hinrichsen and A Kempf. Maximal localization in the presence of minimal uncertainties in positions and in momenta. *J. Math. Phys.* 1996; **37**, 2121-37.

[7] K Nozari and M Karami. Minimal length and generalized Dirac equation. *Mod. Phys. Lett. A* 2005; **20**, 3095-03.

[8] A Kempf and G Mangano. Minimal length uncertainty relation and ultraviolet regularization. *Phys. Rev. D* 1997; **55**, 7909-20.

[9] M Lubo. Quantum minimal length and trans-Planckian photons. *Phys. Rev. D* 2000; **61**, 124009.

[10] LN Chang, D Minic, N Okamura and T Takeuchi. Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations. *Phys. Rev. D* 2002; **65**, 125027.

[11] S Hossenfelder, M Bleicher, S Hofmann, J Ruppert, S Scherer and H Stöcker. Signatures in the Planck regime. *Phys. Lett. B* 2003; **575**, 85-99.

[12] U Harbach, S Hossenfelder, M Bleicher and H Stöcker. Probing the minimal length scale by precision tests of the muon g−2. *Phys. Lett. B* 2004; **584**, 109-13.

[13] M Kober. Gauge theories under incorporation of a generalized uncertainty principle. *Phys. Rev. D* 2010; **82**, 085017.

[14] M Carmeli. Field theory on $R\times S^3$ topology. I: The Klein-Gordon and Schrödinger equations. *Found. Phys.* 1985; **15**, 175-84.

[15] M Carmeli and S Malin. Field theory on $R\times S^3$ topology. II: The Weyl equation. *Found. Phys.* 1985; **15**, 185-91.

[16] M Carmeli and S Malin. Field theory on $R\times S^3$ topology. III: The Dirac equation. *Found. Phys.* 1985; **15**, 1019-29.

[17] M Carmeli and S Malin. Field theory on $R\times S^3$ topology. IV: Electrodynamics of magnetic moments. *Found. Phys.* 1986; **16**, 791-806.

[18] M Carmeli and S Malin. Field theory on $R\times S^3$ topology. V: SU 2-gauge theory. *Found. Phys.* 1987; **17**, 193-200.

[19] M Carmeli and S Malin. Field theory on $R\times S^3$ topology. VI: Gravitation. *Found. Phys.* 1987; **17**, 407-17.

[20] M Carmeli and A Malka. Field theory on $R\times S^3$ topology: Lagrangian formulation. *Found. Phys.* 1990; **20**, 71-110.

[21] H Hassanabadi, S Zarrinkamar and E Maghsoodi. Scattering states of Woods-Saxon interaction in minimal length quantum mechanics. *Phys. Lett. B* 2012; **718**, 678-82.

[22] SA Khorram-Hosseini, S Zarrinkamar and H Panahi. Minimal length Schrödinger equation via factorisation approach. *Pramana* 2019; **92**, 53.

[23] MA Dariescu, C Dariescu and I Gottlieb. Gauge theory of fermions on $R\times S^3$ spacetime. *Found. Phys.* 1995; **25**, 959-63.